National University of Singapore
Microeconomics III, EC4101(L2)
Tutorial 4 Solution: Production and Cost Theory
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1. A firm has production function $y = LK$.

   (a) Derive the conditional input demand functions, $L^*(w, r, y)$ and $K^*(w, r, y)$, i.e., input demands as a function of output and input prices. (Steps: Write the cost-minimization objective function, and the constraint. Then solve the constrained minimization problem and obtain the conditional input demand functions.)

   Answer: The firm solves the following cost-minimization problem subject to a target level of output:

$$\min_{L,K} \; wL + rK \qquad \text{subject to} \qquad F(L, K) = y.$$

   Write the Lagrangean

$$\mathcal{L} = wL + rK + \lambda[y - F(L, K)],$$

   where λ is the shadow-value of the constraint, i.e., the marginal cost given our interpretation of shadow value in lecture.

   The first-order conditions for the above exercise are as follows:

$$\frac{\partial \mathcal{L}}{\partial L} = w - \lambda \frac{\partial F}{\partial L} = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial K} = r - \lambda \frac{\partial F}{\partial K} = 0 \qquad (2)$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = y - F(L, K) = 0. \qquad (3)$$

From (1) and (2), write:

$$
\begin{aligned}
\frac{w}{r} &= \frac{\partial F}{\partial L}\Big/\frac{\partial F}{\partial K} && \text{(isoquant tangent to the isocost line)} \\
\text{i.e.,} \quad \frac{w}{r} &= \frac{K}{L} \\
\text{i.e.,} \quad K &= \frac{w}{r}L,
\end{aligned}
$$

which using in the production function yields

$$y = \frac{w}{r}L^2, \qquad \text{i.e.,} \quad L^* = (\frac{r}{w})^{1/2}y^{1/2}; \quad K^* = (\frac{w}{r})^{1/2}y^{1/2}.$$

(b) Derive the cost function, $C(w, r, y)$.

Answer: The cost function

$$C(w, r, y) = wL^* + rK^* = 2(wr)^{1/2}y^{1/2}.$$

(c) What happens to demand for labor as r changes? What happens to demand for capital as w changes? Are labor and capital substitutes or complements?

Answer:

$$\frac{\partial L^*}{\partial r} = (1/2)\frac{1}{(wr)^{1/2}}y^{1/2} > 0; \qquad \frac{\partial K^*}{\partial w} = (1/2)\frac{1}{(wr)^{1/2}}y^{1/2} > 0.$$

Thus, labor demand would increase in response to an increase in the price of capital, and demand for capital would increase in response to an increase in the price of labor. That is, labor and capital are substitutes. (You may alternatively use the definition of substitutes and complements given in McAfee's book, in terms of the sign of the second derivative cross partial. I'll reward either method, even if the two yield different answers.)

(d) What is the shadow value of output for the cost-minimization problem?

Answer: As indicated in part (a) answer, λ is the shadow value of output i.e., the marginal cost.

(e) Derive the average total cost ($ATC(y)$), average variable cost assuming capital is fixed at some value $\bar{K}$ ($AVC(y;\bar{K})$), and short-run marginal cost ($SRMC(y,\bar{K})$), and long run marginal cost ($LMC(y)$).

Answer:

$$
\begin{aligned}
ATC(y) &= \frac{TC}{y} = \frac{2(wr)^{1/2}y^{1/2}}{y} = \frac{2(wr)^{1/2}}{y^{1/2}}; &\quad (4)\\
AVC(y;\bar{K}) &= \frac{VC}{y} = \frac{wL}{y} = \frac{w(y/\bar{K})}{y} = \frac{w}{\bar{K}}. &\quad (5)
\end{aligned}
$$

To calculate $SRMC(y,\bar{K})$, first calculate

$$
SRTC(y,\bar{K}) = wL + r\bar{K} = \underbrace{w\frac{y}{\bar{K}}}_{\text{variable cost}} + \underbrace{r\bar{K}}_{\text{fixed cost}}.
$$

(Note that in calculating AVC and $SRTC$, we are not determining L by the Lagrangean method. Because K is fixed (at some $\bar{K}$) and only L can be varied, for any given output level the corresponding L is determined uniquely by the production function $y = LK$.) Then,

$$
SRMC(y,\bar{K}) = \frac{\partial SRTC(y,\bar{K})}{\partial y} = \frac{w}{\bar{K}} > 0.
$$

On the other hand,

$$
LRMC(y) = \frac{\partial TC(y)}{\partial y} = (1/2)\cdot 2(wr)^{1/2}y^{-1/2} = (wr)^{1/2}\frac{1}{\sqrt{y}}.
$$

Summary and explanations: Since $LRMC(y) < ATC(y)$, therefore $ATC(y)$ is decreasing in output (see (4)). Also, since $SRMC(y;\bar{K}) = AVC(y;\bar{K})$ and $SRMC$ is constant at $w/\bar{K}$, the average variable cost doesn't change as output changes. So, the thing to remember is that if *marginal*$<$*average* then *average*$\downarrow$; if *marginal*$=$*average* then *average*$--$ (i.e., average stays unchanged); *marginal*$>$*average* then *average*$\uparrow$.

Why is average cost decreasing in output? $F(tL, tK) = t^2(LK) > t(LK) = tF(L, K), \quad \forall t > 1$, so as output expands by varying both labor and capital, per-unit cost of production declines (increasing returns to scale). In the short run, capital cannot be varied, so benefits of economies-of-scale might not be fully realized.

2. A firm has a production function $y = \min\{L, K\}$. Solve questions similar to $(a)-(e)$ as in Question 1 above. Also, does this production function exhibit economies of scale, diseconomies of scale, or CRS?

   (a) Derive the conditional input demand functions, $L^*(w, r, y)$ and $K^*(w, r, y)$, i.e., input demands as a function of output and input prices. (Steps: Write the cost-minimization objective function, and the constraint. Then solve the constrained minimization problem and obtain the conditional input demand functions.)

   Answer: The firm solves the following problem:

$$\min_{L,K} \ wL + rK \qquad \text{subject to} \qquad y = \min\{L, K\}.$$

   Now Lagrangean technique is of no use because the production function is not differentiable. Instead, you need to draw a sample isoquant (say for $y = 1$) which will be L-shaped with vertex along the 45-degree line. So to produce $y = 1$, the firm must employ $L = K = 1$ which yields

$$C(w, r, y = 1) = wL + rK = w + r.$$

To produce any arbitrary $\hat{y}$ units of output, the firm should employ

$$L^* = K^* = \hat{y}.$$

Note that the firm continues to employ L and K at the vertex of the isoquant corresponding to output level, $\hat{y}$.

(b) Derive the cost function, $C(w, r, y)$.

Answer: The cost function is:

$$C(w, r, \hat{y}) = wL^* + rK^* = w\hat{y} + r\hat{y} = (w + r)\hat{y} = \hat{y}C(w, r, 1).$$

(c) What happens to demand for labor as r changes? What happens to demand for capital as w changes? Are labor and capital substitutes or complements?

Answer:

$$\frac{\partial L^*}{\partial r} = 0; \qquad \frac{\partial K^*}{\partial w} = 0;$$

both cost-minimizing labor and capital are unresponsive to input price changes (because, remember, the two inputs must be employed in fixed proportion, one-to-one). Here, labor and capital are *complements*.

(d) What is the shadow value of output for the cost-minimization problem?

Answer: Again, the shadow value of output is nothing but the marginal cost which equals $(w + r)$.

(e) Derive the average total cost ($ATC(y)$), average variable cost assuming capital is fixed at some value $\bar{K}$ ($AVC(y; \bar{K})$), and short-run marginal cost ($SRMC(y, \bar{K})$), and long run marginal cost ($LMC(y)$).

Answer:

$$ATC(y) = \frac{TC}{y} = \frac{(w + r)y}{y} = w + r.$$

$$AVC(y;\bar{K}) = \begin{cases} \frac{wy}{y}\ (= w) & \text{if } 0 < y \leq \bar{K} \\ \text{undefined} & \text{if } y > \bar{K}. \end{cases}$$

Why undefined – output exceeding $\bar{K}$ is not feasible given the fixed amount of capital stock $\bar{K}$. Average variable cost remains constant at the level of w up to $y \leq \bar{K}$.

To calculate $SRMC(y, \bar{K})$, first calculate short-run total costs:

$$SRTC(y;\bar{K}) = wy + r\bar{K} \qquad \text{for } y \leq \bar{K}.$$

(Output $y > \bar{K}$ is not feasible.) So differentiating w..r.t y obtain:

$$SRMC(y;\bar{K}) = \begin{cases} w & \text{if } 0 < y \leq \bar{K} \\ \text{undefined} & \text{if } y > \bar{K}. \end{cases}$$

On the other hand,

$$LRMC(y) = LRAC(y) = w + r > 0.$$

(The technology is one of <u>constant returns to scale</u> variety ($F(tL, tK) = \min\{tL, tK\} = t\min\{L, K\} = tF(L, K), \quad \forall t > 1$.) As output expands by varying both labor and capital, per-unit cost of production remains constant. In the short run, output can be expanded one-to-one only by varying labor (up to $\bar{K}$ units), so short-run marginal cost is constant and equal to w.

You should also compare $SRMC(y;\bar{K})$ and $LRMC(y)$; why $SRMC < LRMC$ at $y \leq \bar{K}$? The reasons is, the shadow value of capital is negative for $y < \bar{K}$, that is, if the firm had the option it would have shed off some capital and save on its costs. For output levels below $\bar{K}$, expanding output is relatively cheap (compared to the long run) as the firm only needs to employ more workers (with excess capital available at zero extra cost).

The above result that $SRMC < LRMC$ (at $y \leq \bar{K}$) differs from the traditional view that in the long run incremental costs should be lower than in the short run due to greater flexibility.

3. In Question 1, assume a given fixed price, p, for the produced output y. Now if you try to maximize profit, what would your solution be? Can the maximized profit be strictly positive? Explain.

Answer: There cannot be any solution to the profit maximization problem. Why? Suppose there is a solution (L^*, K^*) that maximizes profit. So,

$$\pi(L^*, K^*) = p(L^* K^*) - [wL^* + rK^*] \geq 0.$$

Now by scaling up (L^*, K^*) to $(2L^*, 2K^*)$, the new profit level becomes

$$\begin{aligned} \pi(2L^*, 2K*) &= p((2L^*) \cdot (2K^*)) - [w(2L^*) + r(2K^*)] \\ &= 2p(L^* K^*) + 2p(L^* K^*) - 2[wL^* + rK^*] \\ &= 2p(L^* K^*) + 2\pi(L^*, K^*) \\ &> \pi(L^*, K^*). \end{aligned}$$

So by employing $(2L^*, 2K^*)$, the firm is able to make more profit than (L^*, K^*), contradicting that (L^*, K^*) was profit-maximizing.

In general, for **increasing returns to scale** technologies no solution exists to the profit maximization problem in the long run for the firm: if some (L, K) achieve positive profit then by scaling up operations the profit will be even higher, and so on. For **constant returns to scale** technology, the only solution to profit maximization (in the long run) is if maximized profit equals **zero**; here you cannot have positive profit.