EC4101
Microeconomics Analysis III
(Group 2)

Topic 5

Game Theory

Game Theory

- Non-strategic setting
  - Payoff independent on actions of another person
  - consumers are price taker (we have assumed)
- strategic setting
  - Payoff may depend on actions of another person

HISTORY OF GAME THEORY

History of Game Theory: 1838

- Augustin Cournot
- “Researches into the Mathematical Principles of the Theory of Wealth” Chapter 7
- On Competition of Producer
- Study special case of duopoly and applies solution concept similar to Nash equilibrium

History of Game Theory: 1871

- Charles Darwin
- “The Descent of Man, and Selection in Relation to Sex” 1st edition
- First (implicitly) game theoretic argument in evolutionary biology.
- Natural section will act to equalize the sex ratio.
  - Births(females) < Births(Male)
  - Mating Prospect(newborn female) > Mating Prospect(newborn male)
  - Newborn female expect to have more offspring.
  - Parents genetically disposed to produce females tend to have more than the average numbers of grandchildren
  - Genes for female-producing tendencies spread, and female births become commoner.
  - As the 1:1 sex ratio is approached, the advantage associated with producing females dies away.
  - The same reasoning holds if males are substituted for females throughout. Therefore 1:1 is the equilibrium ratio.

History of Game Theory: 1881

- Francis Ysidro Edgeworth
- “Mathematical Psychics: An Essay on the Application of Mathematics to the Moral Sciences. “
- Proposed the contract curve as a solution to the problem of determining the outcome of trading between individuals.
  – two commodities and two types of consumers
  – contract curve shrinks to the set of competitive equilibria as the number of consumers of each type becomes infinite.
  – The concept of the core is a generalization of Edgeworth's contract curve.

History of Game Theory: 1913

- Ernst Zermelo
- “Uber eine Anwendung der Mengenlehre auf die Theorie des Schachspiels”
- The first 'theorem' of game theory
- Asserts that in chess either white can force a win, or black can force a win, or both sides can force at least a draw.
- Referred to as Zermelo's Theorem.

History of Game Theory: 1928

- John von Neumann
- Zur Theorie der Gesellschaftsspiele
- Introduced the extensive form of a game
- Minimax theorem: every two- person zero-sum game with finitely many pure strategies when mixed strategies are admittedhas precisely one individually rational payoff vector.

History of Game Theory: 1944

- John von Neumann and Oskar Morgenstern
- “Theory of Games and Economic Behavior”
- Two-person zero sum theory
- Transferable utility (TU)
- von Neumann-Morgenstern stable sets
- Axiomatic utility theory

History of Game Theory 1950

- Melvin Dresher and Merrill Flood
- At the Rand Corporation, the experiment which introduced the game now known as the **Prisoner's Dilemma**.
- A. W. Tucker write the famous story associated
- Howard Raiffa independently conducted, unpublished, experiments with the Prisoner's Dilemma.

History of Game Theory 1950-1953

- John Nash
- Equilibrium Points in N- Person Games (1950) and Non-cooperative Games (1951):
  - proved existence of a strategic equilibrium for non-cooperative games-**the Nash equilibrium**
  - proposed the "Nash program": the study of cooperative games via their reduction to non-cooperative form.
- The Bargaining Problem (1950) and Two-Person Cooperative Games (1953)
  - founded axiomatic bargaining theory
  - existence of the Nash bargaining solution
  - first execution of the Nash program

History of Game Theory 1952-53

- L. S. Shapley
- Notes on the N-Person Game III: Some Variants of the von-Neumann-Morgenstern Definition of Solution
- D. B. Gillies
- Some Theorems on N-Person Games
- Core as a general solution concept
- The core is the set of allocations that cannot be improved upon by any coalition.

History of Game Theory 1953

- Lloyd Shapley
- A Value for N-Person Games characterised, by a set of axioms,
- **Shapley Value**: a solution concept that associates with each coalitional game, a unique outcome

History of Game Theory Late 1950

- Authorship is obscure
- Folk Theorem: equilibrium outcomes in an infinitely repeated game coincide with the feasible and strongly individually rational outcomes of the one-shot game on which it is based.

History of Game Theory 1965

- R. Selten
- "Spieltheoretische Behandlung eines Oligopolmodells mit Nachfragetraegheit"
- (subgame) perfect equilibria: Idea of refinements of the Nash equilibrium with the concept of

History of Game Theory 1967-68

- John Harsanyi
- Games with Incomplete Information Played by 'Bayesian' Players, Parts I, II and III
- constructed the theory of games of incomplete information
- theoretical groundwork for information economics

Four Different Games

- Complete Information Static Game
  - Normal Form Game: Dominance, Nash Equilibrium
  - Players move simultaneously
- Complete Information Dynamic Game
  - Extensive Form Game: Subgame Perfect Equilibrium
  - Players are not moving simultaneously
  - Important Extension: Repeated Game
- Incomplete Information Static Game
  - Bayesian Game: Bayesian Nash Equilibrium
- Incomplete Information Dynamic Game
  - Signaling Game: Perfect Bayesian Equilibrum

Static Games of complete information

Normal Form Game

Strict Dominance

Nash Equilibrium

Normal Form Game

- Normal Form Game $G=(N,S,u)$
  - Players $N=\{1,2,\ldots,N\}$
    - May be individuals, firms, countries, etc.
    - Have the ability to choose from among a set of possible actions
  - Strategies $S = (S_1,S_2,\ldots,S_N)$
    - S_i is the set of strategies open to player i
    - s_i is the strategy chosen by player i, $s_i \in S_i$
  - Payoffs $u=(u_1,u_2,\ldots,u_N)$
    - $u_i\!:S\rightarrow\Re$: payoff function mapping form strategies to number
    - $u_1(s_1,s_2)$: 1's payoff if 1 follows s_1 and 2 follows s_2
    - $u_2(s_2,s_1)$: 2's payoff if 1 follows s_1 and 2 follows s_2

Normal Form Game

- Outcome is a list of strategies $(s_1, s_2, \ldots, s_N)$
- Goal: predict possible outcomes of the game
- We focus on two concepts:
  - Strict Dominance
  - Nash equilibrium

Example: Prisoner’s Dilemma

- Two suspects are arrested for a crime
- Police wants to extract a confession:
  - If you fink on your companion, but your companion doesn’t fink on you, you get a one-year sentence and your companion gets a four-year sentence”
  - “If you both fink on each other, you will each get a three-year sentence”
  - “If neither finks, we will get tried for a lesser crime and each get a two-year sentence”

FIGURE 8.1

Normal Form for the Prisoners' Dilemma

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$u_1 = 1, u_2 = 1$	$u_1 = 3, u_2 = 0$
	Silent	$u_1 = 0, u_2 = 3$	$u_1 = 2, u_2 = 2$

Strict Dominance

- Strategy s_i is (**strictly) dominated** if for some s'_i

$$u_i(s'_i, s_{-i}) > u_i(s_i, s_{-i}) \text{ for all } s_{-i}$$

- Strategy s_i is a (**strictly) dominant strategy** if it strictly dominates other strategy in S_i

Elimination of Dominated Strategy

- Rationality implies no player would choose strictly dominated strategy
- Prediction should exclude those outcomes!
- In prisoner dilemma, Finking is a dominant strategy for both players
- Hence, both are finking!

Iterated Elimination

- We can repeat the elimination: iterated elimination of strictly dominated strategy (i.e.s.d.s)
- However, it may not be possible to eliminate any: Rock-Paper-Scissor game

Rock, Paper, Scissors

(a) Normal form

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissors	−1, 1	1, −1	0, 0

(b) Underlining procedure

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	0, 0	$-1, \underline{1}$	$\underline{1}, -1$
Paper	$\underline{1}, -1$	0, 0	$-1, \underline{1}$
Scissors	$-1, \underline{1}$	$\underline{1}, -1$	0, 0

Nash Equilibrium

- Immune to unilateral deviation
- Best response
  - s_i is the best response for i to rivals' strategies s_{-I}

$$u_i(s_i,s_{-i}) \geq u_i(s'_i,s_{-i}) \text{ for all } s'_i \in S_i$$

  - denoted as $s_i \in BR_i(s_{-i})$
- Nash equilibrium: strategy profile $(s_1,s_2,\ldots s_n)$
  - s_i is a best response to s_{-i} or

$$s^*_i \in BR_i(s^*_{-i})$$

Nash Equilibrium and Iterated Elimination

- Nash equilibrium survives iterated elimination of strictly dominated strategies
- If the elimination process ends with unique outcome, it is the Nash equilibrium

FIGURE 8.2

Underlining Procedure in the Prisoners' Dilemma

		Suspect 2	
		Fink	Silent
Suspect 1	Fink	$\underline{u_1 = 1}, \underline{u_2 = 1}$	$\underline{u_1 = 3}, u_2 = 0$
	Silent	$u_1 = 0, \underline{u_2 = 3}$	$u_1 = 2, u_2 = 2$

FIGURE 8.3

Normal Form for the Battle of the Sexes

		Player 2 (Husband)	
		Ballet	Boxing
Player 1 (Wife)	Ballet	2, 1	0, 0
	Boxing	0, 0	1, 2

FIGURE 8.4

Underlining Procedure in the Battle of the Sexes

		Player 2 (Husband)	
		Ballet	Boxing
Player 1 (Wife)	Ballet	$\underline{2}, \underline{1}$	0, 0
	Boxing	0, 0	$\underline{1}, \underline{2}$

Existence of Pure Strategy NE

- Under some conditions, it exists. (See Extension Slides)
- Not exist for Rock-paper-scissor game

FIGURE 8.6

Rock, Paper, Scissors

(a) Normal form

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	0, 0	−1, 1	1, −1
Paper	1, −1	0, 0	−1, 1
Scissors	−1, 1	1, −1	0, 0

(b) Underlining procedure

Player 1 \ Player 2	Rock	Paper	Scissors
Rock	0, 0	−1, $\underline{1}$	$\underline{1}$, −1
Paper	$\underline{1}$, −1	0, 0	−1, $\underline{1}$
Scissors	−1, $\underline{1}$	$\underline{1}$, −1	0, 0

Mixed Strategy

- M Pure strategy $S_i = \{s^1{}_i,\ldots s^m{}_i,\ldots,s^M{}_i\}$
- Mixed strategy: probability distribution over M actions

$$\sigma_i = (\sigma^1{}_i,\ldots,\sigma^m{}_i,\ldots,\sigma^M{}_i)$$

  – $\sigma^m{}_i$ indicates the probability of player i playing action $s^m{}_i$
    - $0 \le \sigma^m{}_i \le 1$
    - $\sigma^1{}_i+\ldots+\sigma^m{}_i+\ldots+\sigma^M{}_i = 1$

- Suppose the wife chooses mixed strategy (1/9,8/9) and the husband chooses (4/5,1/5)
- The wife's expected payoff is

$$\left(\frac{1}{9}\right)\left(\frac{4}{5}\right)u_1(\text{ballet,ballet})+\left(\frac{1}{9}\right)\left(\frac{1}{5}\right)u_1(\text{ballet,boxing})$$

$$+\left(\frac{8}{9}\right)\left(\frac{4}{5}\right)u_1(\text{boxing,ballet})+\left(\frac{8}{9}\right)\left(\frac{1}{5}\right)u_1(\text{boxing,boxing})$$

$$=\left(\frac{1}{9}\right)\left(\frac{4}{5}\right)(2)+\left(\frac{1}{9}\right)\left(\frac{1}{5}\right)(0)+\left(\frac{8}{9}\right)\left(\frac{4}{5}\right)(0)+\left(\frac{8}{9}\right)\left(\frac{1}{5}\right)(1)=\frac{16}{45}$$

EXAMPLE 8.3 Expected Payoffs in the Battle of the Sexes

- Suppose the wife chooses mixed strategy (*w*,1-*w*) and the husband chooses (*h*,1-*h*)
  - The wife plays ballet with probability *w* and the husband with probability *h*
- Her expected payoff becomes

$$(w)(h)U_1(\text{ballet,ballet})+(w)(1-h)U_1(\text{ballet,boxing})+$$

$$+(1-w)(h)U_1(\text{boxing,ballet})+$$

$$+(1-w)(1-h)U_1(\text{boxing,boxing})=$$

$$=(w)(h)(2)+(w)(1-h)(0)+(1-w)(h)(0)+(1-w)(1-h)(1)$$

$$=1-h-w+3hw$$

EXAMPLE 8.3 Expected Payoffs in the Battle of the Sexes

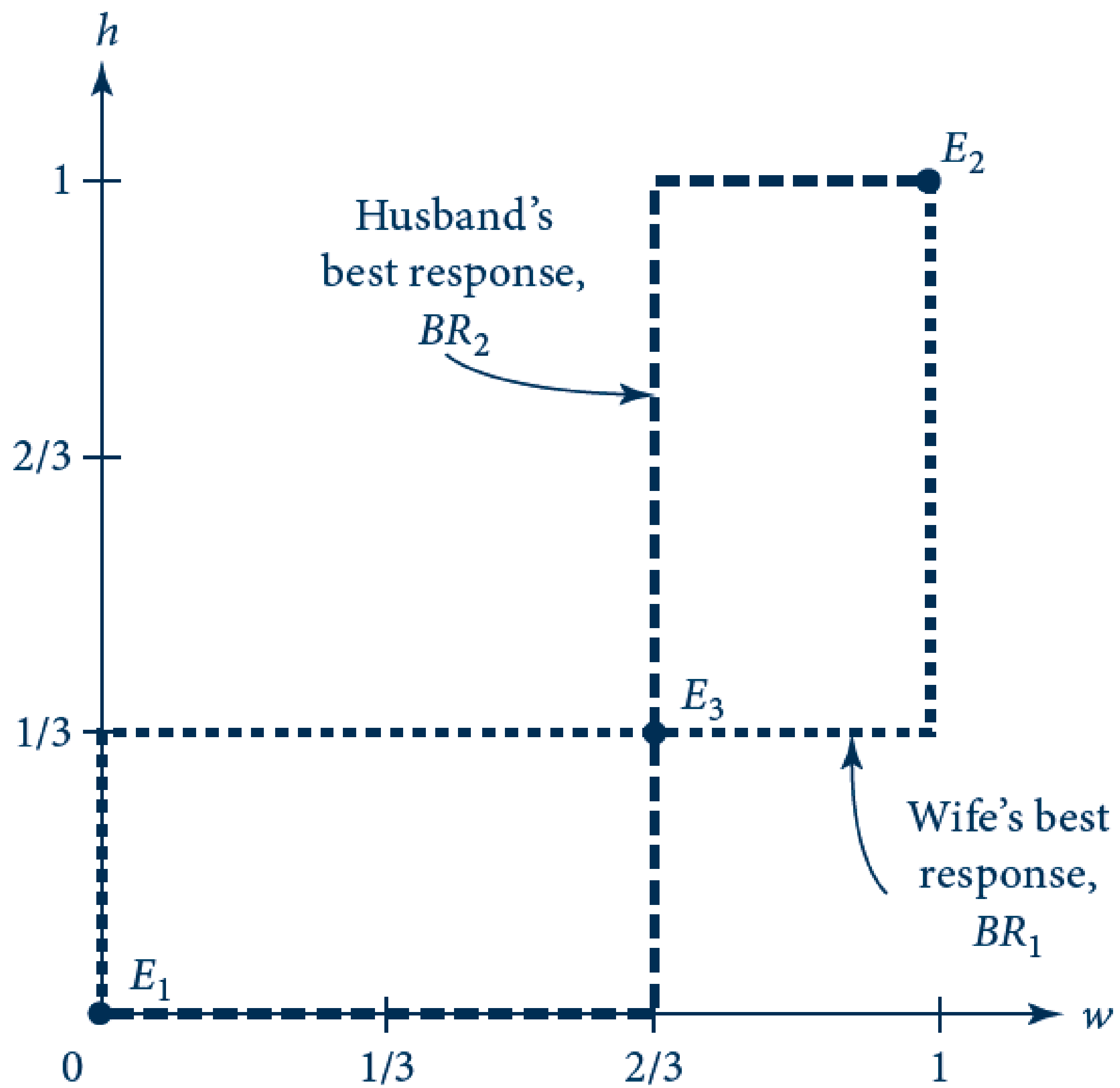
- The wife's best response depends on h
  - If $h < 1/3$, she should set $w = 0$
  - If $h > 1/3$, she should set $w = 1$
  - If $h = 1/3$, her expected payoff is the same no matter what value of w she chooses
- The husband's expected payoff is

$$2 - 2h - 2w + 3hw$$

  - when $w < 2/3$, he should set $h = 0$
  - when $w > 2/3$, he should set $h = 1$
  - when $w = 2/3$, his expected payoff is the same no matter what value of h he chooses

FIGURE 8.7

Nash Equilibria in Mixed Strategies in the Battle of the Sexes



Ballet is chosen by the wife with probability w and by the husband with probability h. Players' best responses are graphed on the same set of axes. The three intersection points E_1, E_2, and E_3 are Nash equilibria. The Nash equilibrium in strictly mixed strategies, E_3, is $w^*=2/3$ and $h^*=1/3$.

Mixed Strategy Nash Equilibrium

- Definition is the same
- Given other players' strategy, any player will only employ mix over pure strategies because they give the same payoff
  - In rock-paper-scissors, I mix because you mix!
- Mixing with strictly dominated strategy is suboptimal
  - No need to calculate them

Existence of Nash equilibrium

- Kakutani's fixed point theorem (Nash 1950): existence of mixed strategy equilibrium under certain conditions

Continnum of Actions

- Strategy is not finite but infinite
- Tragedy of Commons:
  - Write down the payoff for each player as a function of all players' actions
  - Compute the first-order condition associated with each player's payoff maximum
    - Equation - can be rearranged into the best response of each player as a function of all other players' actions
    - Solve the system of n equations for the n unknown equilibrium actions

- The "Tragedy of the Commons"
  - Describes the overuse that arises when scarce resources are treated as common property
  - Two herders decide how many sheep to graze on the village commons
  - The commons is quite small and can rapidly succumb to overgrazing
  - q_i = the number of sheep chosen by herder i
  - Per-sheep value of grazing on the commons is

$$v(q_1,q_2) = 120 - (q_1 + q_2)$$

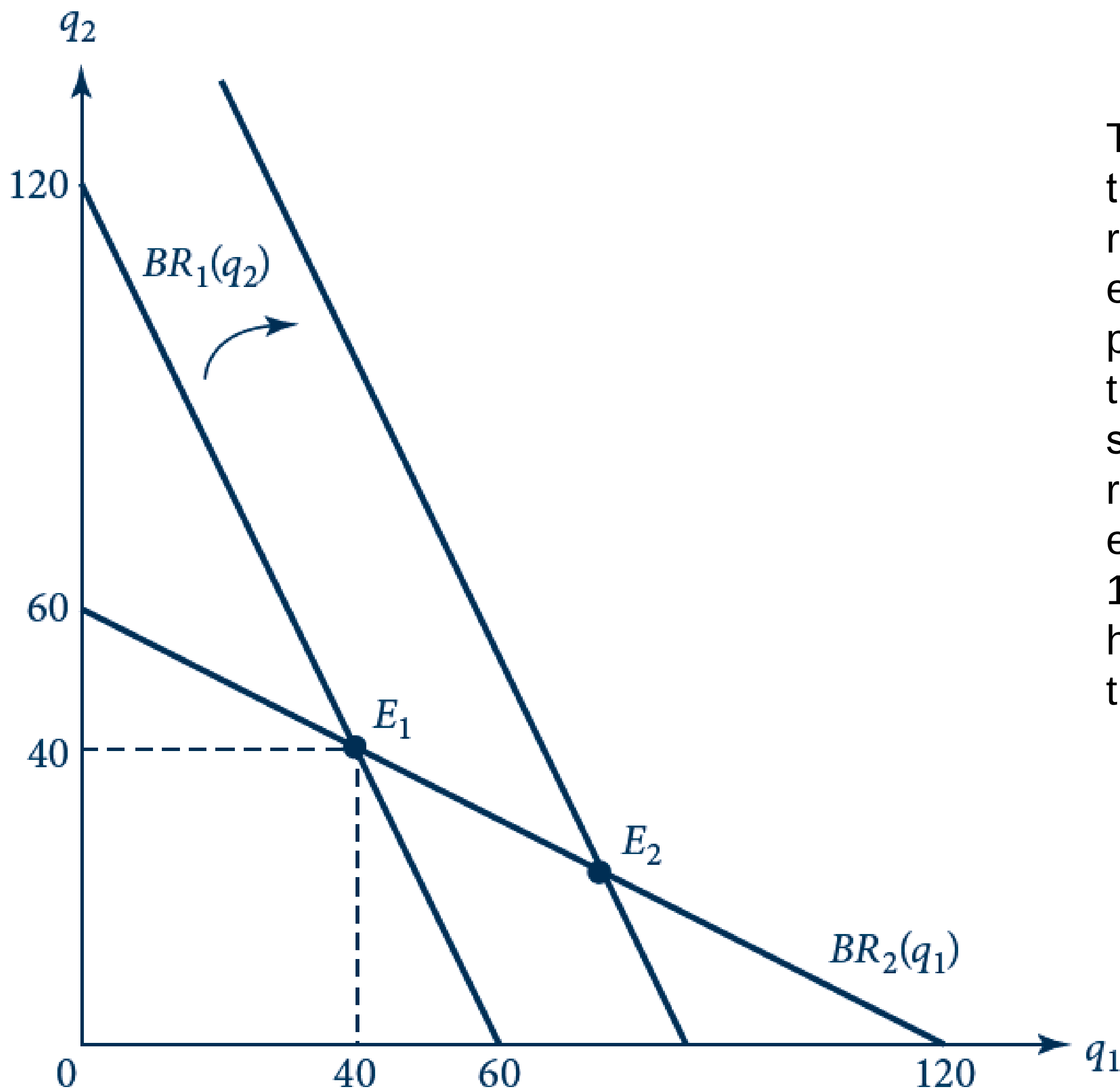
- The "Tragedy of the Commons"
  - The normal form is a listing of the herders' payoff functions

$$u_1(q_1,q_2) = q_1 v(q_1,q_2) = q_1(120 - q_1 - q_2)$$
$$u_2(q_1,q_2) = q_2 v(q_1,q_2) = q_2(120 - q_1 - q_2)$$

- Solve for the Nash equilibrium
  - Solve herder 1's maximization problem and get his best-response function: $q_1=60-q_2/2=BR_1(q_2)$
  - Solve herder 2's maximization problem and get his best-response function: $q_2=60-q_1/2=BR_2(q_1)$
  - The Nash equilibrium: $q^*_1=q^*_2=40$, payoff=1,600

FIGURE 8.8

Best-Response Diagram for the Tragedy of the Commons



The intersection, E_1, between the two herders' best responses is the Nash equilibrium. An increase in the per-sheep value of grazing in the Tragedy of the Commons shifts out herder 1's best response, resulting in a Nash equilibrium E_2 in which herder 1 grazes more sheep (and herder 2, fewer sheep) than in the original Nash equilibrium.

EXAMPLE 8.5 Tragedy of the Commons

- Suppose the per-sheep value of grazing rises for herder 1
  - Would result in more sheep for herder 1 and fewer for herder 2
- The Nash equilibrium is not the best use of the commons
  - If both herders grazed 30 sheep each, their payoffs would rise
- Solving a joint-maxizimization problem will lead to the higher payoffs

Dynamic Games of Complete Information

Extensive Form

Subgame Perfect Equilibrium

Repeated Game

Extensive Form Game

- Players not move simultaneously
  - Order of players move is important
- Might see how others have played
  - Information environment

Formal Definition

- Extensive Form Game: $G=\{N,T,I,n,A,u,P\}$
- Set of Players: $N=\{1,2,\ldots,N\}$
- Set of Nodes: T;
  - Terminal node: Z
  - Decision node (non-terminal node) $t\in T\backslash Z$:
    - Who moves at node t: i(t) where $i:T\rightarrow N$
    - Set of Action at node t: A(t)
    - Successor node: n(t,a) where $n: (T\backslash Z)\times A\rightarrow T$
- Payoff functions: $u_i:Z\rightarrow\Re$
- Information Set P(t): set of nodes player i(t) knows it is possible
  - Partition
  - $t'\in P(t)$ implies $i(t')=i(t)$, $A(t')=A(t)$ and $P(t')=P(t)$

Action and Strategy

- Action: choice at a decision node
- Strategy: selected action at every possible decision node (even it is not reached according to the play)

Information Set

- Players might not exactly know which decision node they are in
- Information set: set of decision nodes that players cannot tell exactly where they are in
- Can embed any normal-form game!

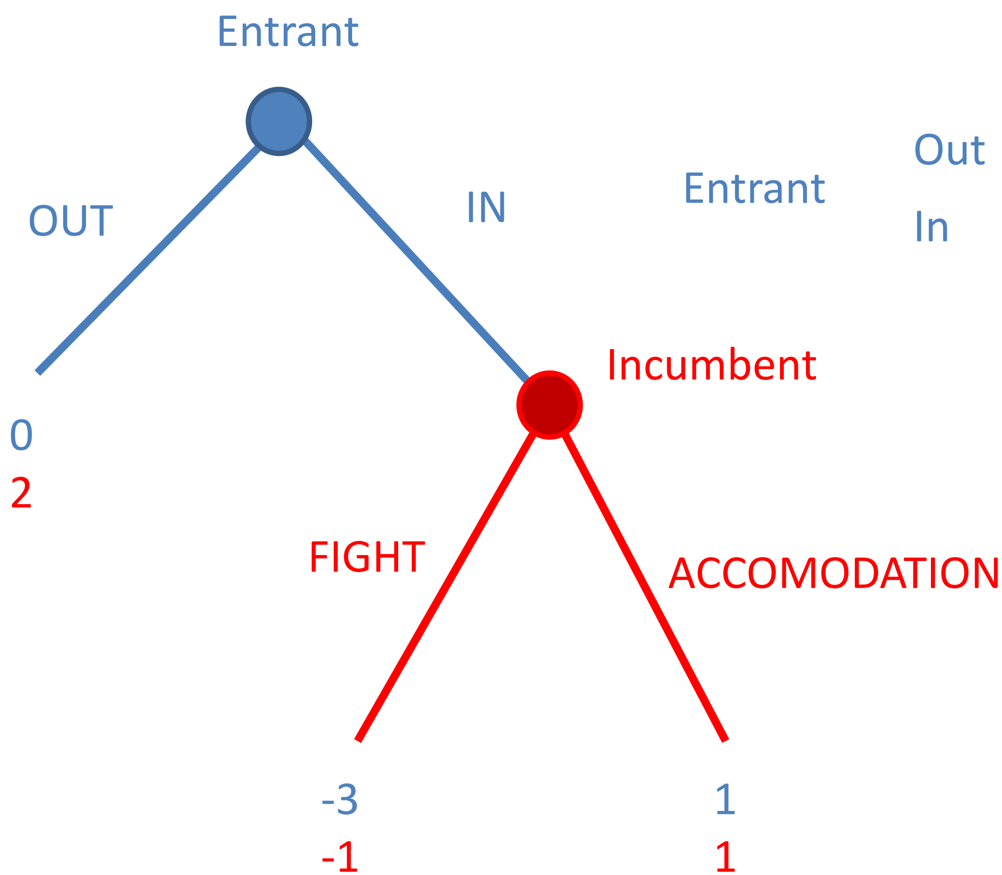
Convert to Normal Form

- Convert back to Normal Form
- Apply “Nash Equilibrium”
- Do we have reasonable result?

Example: Predation Game

- Famous example by Selten
- Two players: entrant and incumbent:
- 1st) An entrant decides to in or out
- 2nd) Incumbent fight or accommodate if in
- No entry: U(entrant)=0; U(Incumbent)=2
- Entry: Fight is costly for both
  - Fight: U(entrant)=-1; U(Incumbent)=-3
  - Accommodate: U(entrant)=1; U(Incumbent)=1

Predation Game



		Incumbent	
		Fight if in	Accommodate if in
Entrant	Out	0,2	0,2
	In	-3,-1	1,1

Predation Game

- Two Nash Equilibria:
  – (out, fight) and (in, accommodate)
- However, (out, fight) is based on empty threat.
  – Out because you want to fight
  – But you will not fight if it is actually in
- Nash Equilibrium is NOT a good concept!

Principle of sequential rationality

- Strategy should specify optimal actions at *every* possible node in the tree
- Reference: dynamic programming

Backward Induction

- If every player are sure about which decision node they are, it becomes a sequential decision problem
- From dynamic programming, we can use backward induction
  - Solve from the last decision node

Subgame

- A subgame
  - A part of the extensive form beginning with a decision node and including everything to the right of it
- A proper subgame
  - Starts at a decision node not connected to another in an information set

Subgame Perfect Equilibirum

- Subgame perfect Equilibrium (SPE) means it is NE in every proper subgame
  - Is always a Nash equilibrium
  - Sequential Rationality: rules out any empty threat in a sequential game

Sequential Battle of Sexes

- Suppose in battle of sexes: wife chooses first
  - And the husband observes her choice before making his
  - Her possible strategies haven't changed
  - His possible strategies have expanded
    - For each of his wife's actions, he can choose one of two actions

TABLE 8.1

Husband's contingent strategies

Contingent Strategy	Written in Conditional Format
Always go to the ballet	(ballet | ballet, ballet | boxing)
Follow his wife	(ballet | ballet, boxing | boxing)
Do the opposite	(boxing | ballet, ballet | boxing)
Always go to boxing	(boxing | ballet, boxing | boxing)

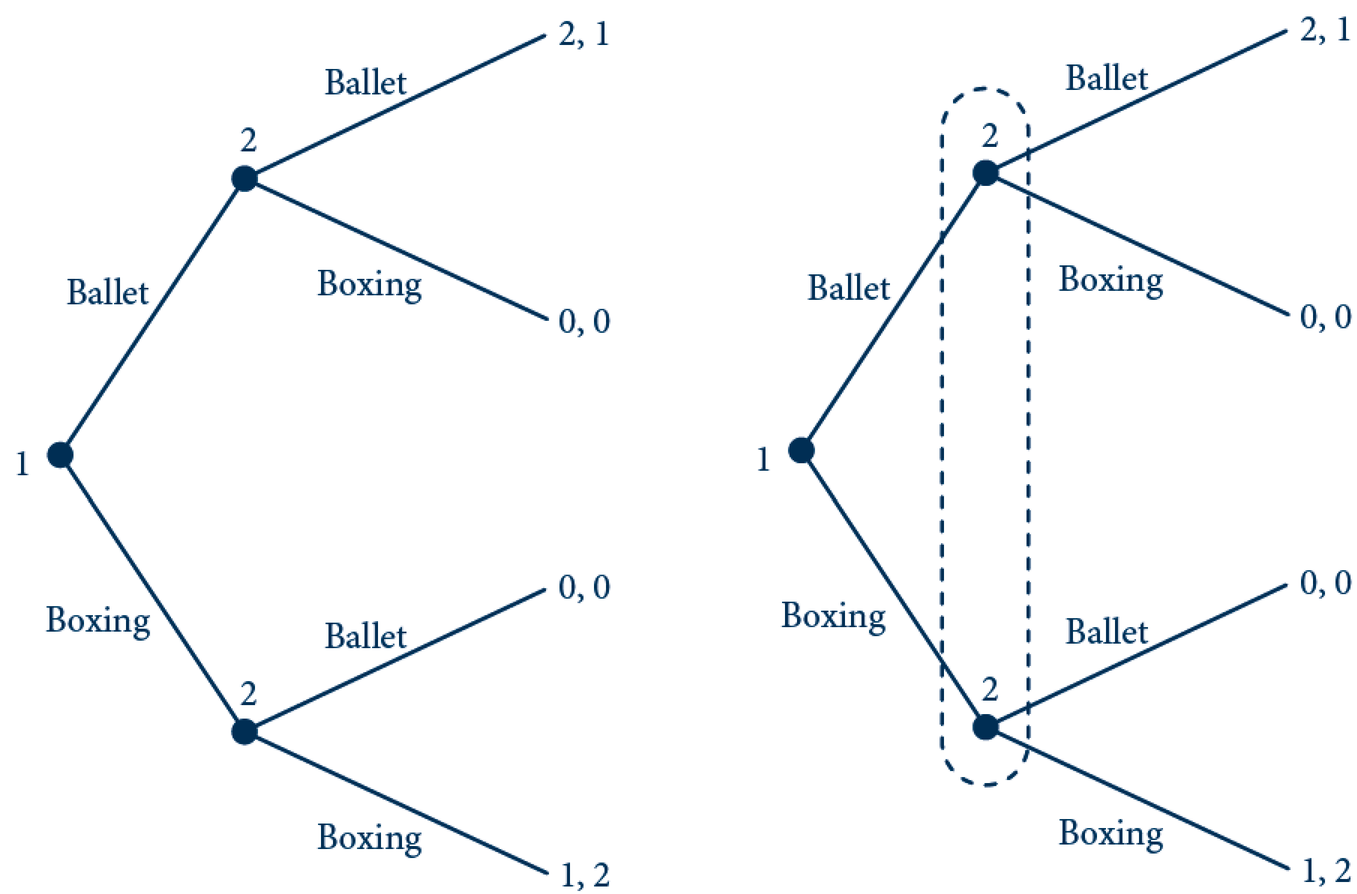
FIGURE 8.9

Normal Form for the Sequential Battle of the Sexes

<table>
<tr><th></th><th></th><th colspan="4">Husband</th></tr>
<tr><th></th><th></th><th>(Ballet | Ballet
Ballet | Boxing)</th><th>(Ballet | Ballet
Boxing | Boxing)</th><th>(Boxing | Ballet
Ballet | Boxing)</th><th>(Boxing | Ballet
Boxing | Boxing)</th></tr>
<tr><td rowspan="2">Wife</td><td>Ballet</td><td>2, 1</td><td>2, 1</td><td>0, 0</td><td>0, 0</td></tr>
<tr><td>Boxing</td><td>0, 0</td><td>1, 2</td><td>0, 0</td><td>1, 2</td></tr>
</table>

FIGURE 8.10

Extensive Form for the Battle of the Sexes



(a) Sequential version (b) Simultaneous version

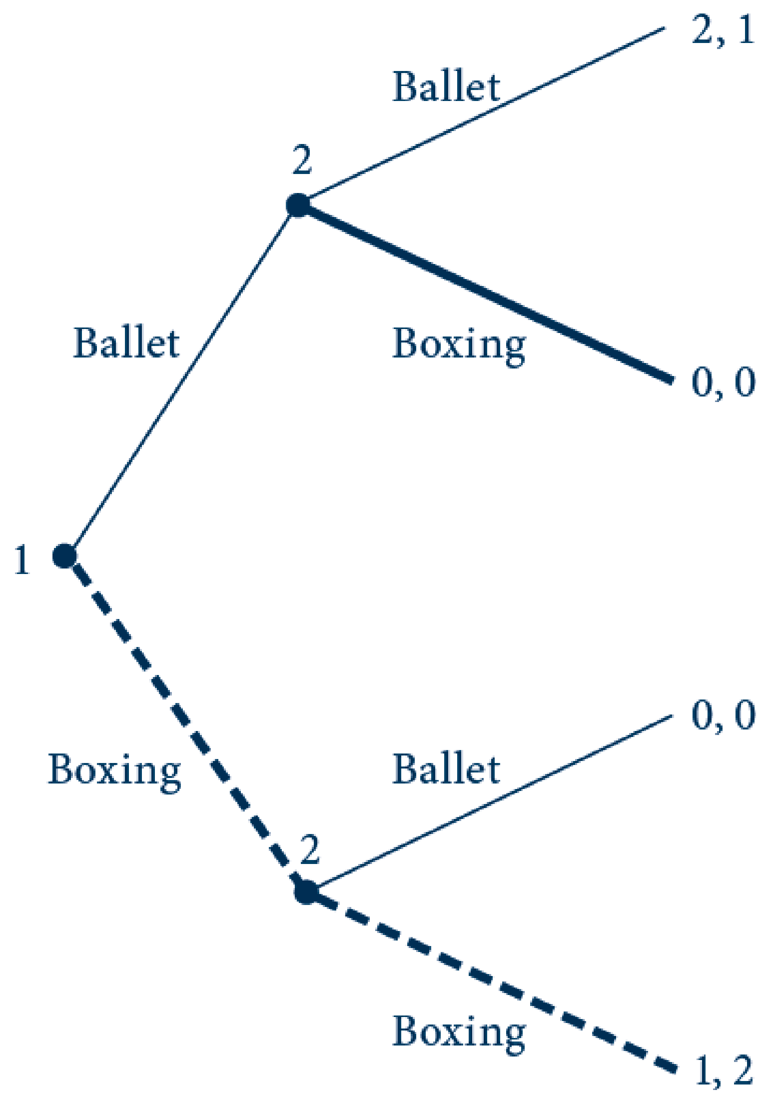
In the sequential version (a), the husband moves second, after observing his wife's move. In the simultaneous version (b), he does not know her choice when he moves, so his decision nodes must be connected in one information set.

pure-strategy Nash equilibria

- Three pure-strategy Nash equilibria
  1. Wife plays ballet, husband plays (ballet | ballet, ballet | boxing)
  2. Wife plays ballet, husband plays (ballet | ballet, boxing | boxing)
  3. Wife plays boxing, husband plays (boxing | ballet, boxing | boxing)

FIGURE 8.11

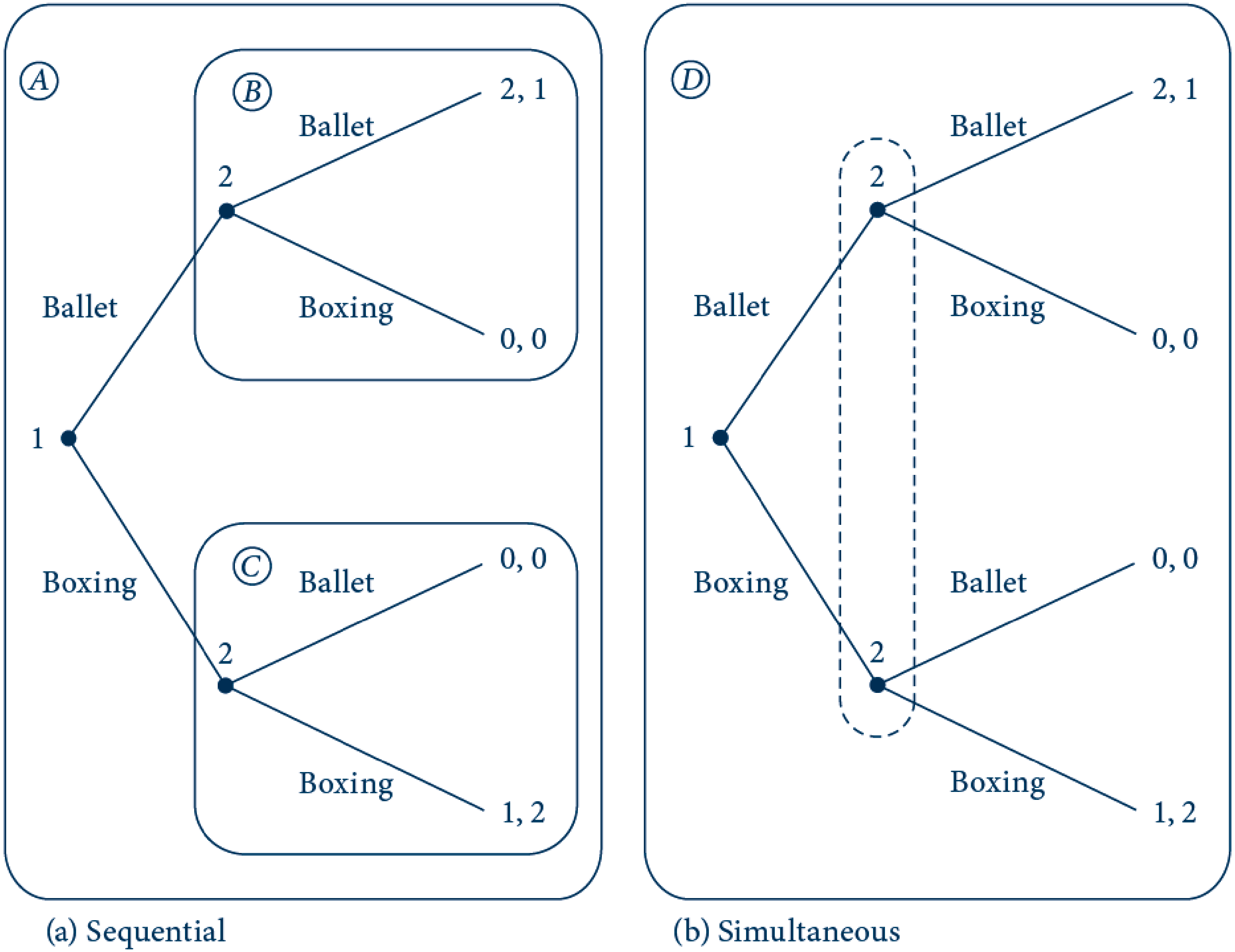
Equilibrium Path



In the third of the Nash equilibria listed for the sequential Battle of the Sexes, the wife plays boxing and the husband plays (boxing | ballet, boxing | boxing), tracing out the branches indicated with thick lines (both solid and dashed). The dashed line is the equilibrium path; the rest of the tree is referred to as being ‘‘off the equilibrium path.’’

FIGURE 8.12

Proper Subgames in the Battle of the Sexes



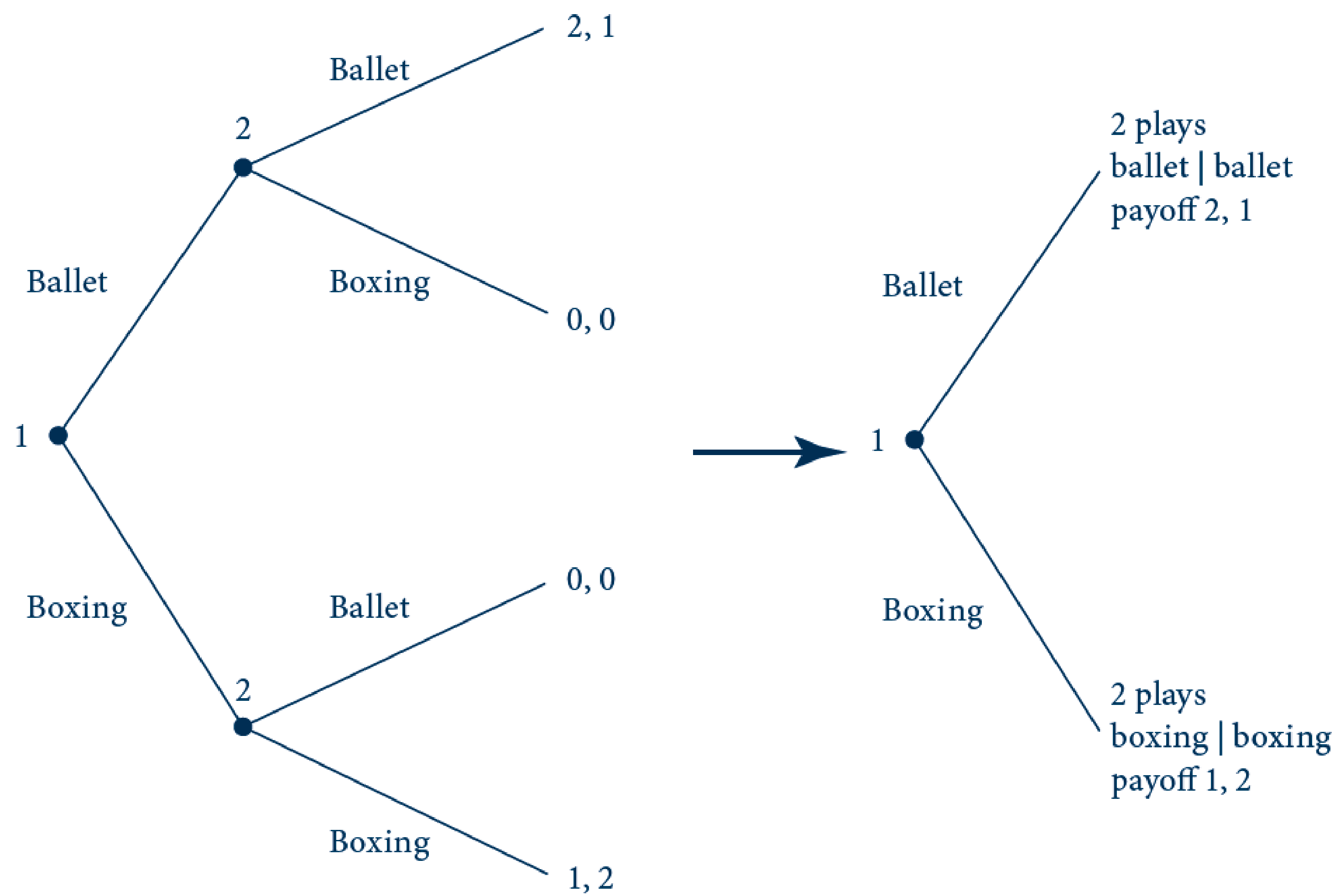
The sequential version in (a) has three proper subgames, labeled A, B, and C. The simultaneous version in (b) has only one proper subgame: the whole game itself, labeled D.

Backward Induction

- A shortcut for finding the perfect-subgame equilibrium directly
- Working backwards from the end of the game to the beginning
  - Compute the Nash equilibria for the bottommost subgames at the husband's decision nodes
  - Substitute his equilibrium strategies for the subgames themselves
  - The resulting game is a simple decision problem for the wife

FIGURE 8.13

Applying Backward Induction



The last subgames (where player 2 moves) are replaced by the Nash equilibria on these subgames. The simple game that results at right can be solved for player 1's equilibrium action.

Repeated Game

- Prisoner's Dilemma
- Always defect: seems puzzling
  - But what stop you being naughty if this is the only chance you are with your partner?
- What if this kind of game is repeated?
  - You and your companion have future interactions
- Repeated Game: stage game is played several times

Finitely Repeated Game

- Does NOT help
- Selten's theorem:
  - For any stage game with a unique Nash equilibrium
  - The unique subgame-perfect equilibrium of the finitely repeated game
  - Involves playing Nash equilibrium every period

Finitely Repeated Game

- If the stage game has multiple Nash equilibria
  – It may be possible to achieve cooperation in a finitely repeated game
- Players can use **trigger strategies** to maintain cooperation
    - Threaten to play the Nash equilibrium that yields a worse outcome for the player who deviates

Infinitely Repeated Game

- Folk Theorem: almost everything if players are patient
- Grim trigger: players revert to the harshest punishment possible
- tit-for-tat: only one round of punishment for cheating

Prisoners' Dilemma

- Players follow a trigger strategy
  - If both players are silent every period, the payoff over time would be

$$V^{eq} = 2 + 2\delta + 2\delta^2 + \ldots = 2/(1-\delta)$$

  - If a player deviates and then the other finks every period, that player's payoff is

$$V^{dev} = 3 + 1\delta + 1\delta^2 + \ldots = 3 + \delta/(1-\delta)$$

  - Trigger strategies form a perfect-subgame equilibrium: $V^{eq} \geq V^{dev}$, so $\delta \geq 1/2$

Static Game of Incomplete Information

Bayesian Game

Bayesian Nash Equilibrium

Incomplete Information

- Players that lack full information of the game
- Difficult to model
- Selten: games of imperfect information
  - players have different types
  - players know their own types but not others'
  - Players have belief over types of others

Types

- Strategy and payoff are depend on types
- Strategy: S_i:T$\rightarrow\Re$
- Payoff: u_i:S$\times$T$\rightarrow\Re$

Bayesian Game

- Bayesian Game: G=(N,S,u,T,P) where
  - Players: N={1,2,…,N}
  - Strategy: $S = (S_1, S_2, \ldots, S_N)$
    - Need to specify strategy for each type: $S_i: T \rightarrow \Re$
  - Payoff: $u = (u_1, u_2, \ldots, u_N)$
    - Payoff depend on types: $u_i: S \times T \rightarrow \Re$
  - Type: $T = (T_1, T_2, \ldots, T_N)$
    - Player 1's type: $T_1 = \{t_{1a}, t_{1b}, t_{1c}, \ldots\}$ (if finite)
  - Belief: $P = (P_1, P_2, \ldots, P_N)$
    - Probability on other peoples type: $P_i: T \rightarrow [0,1]$

FIGURE 8.14

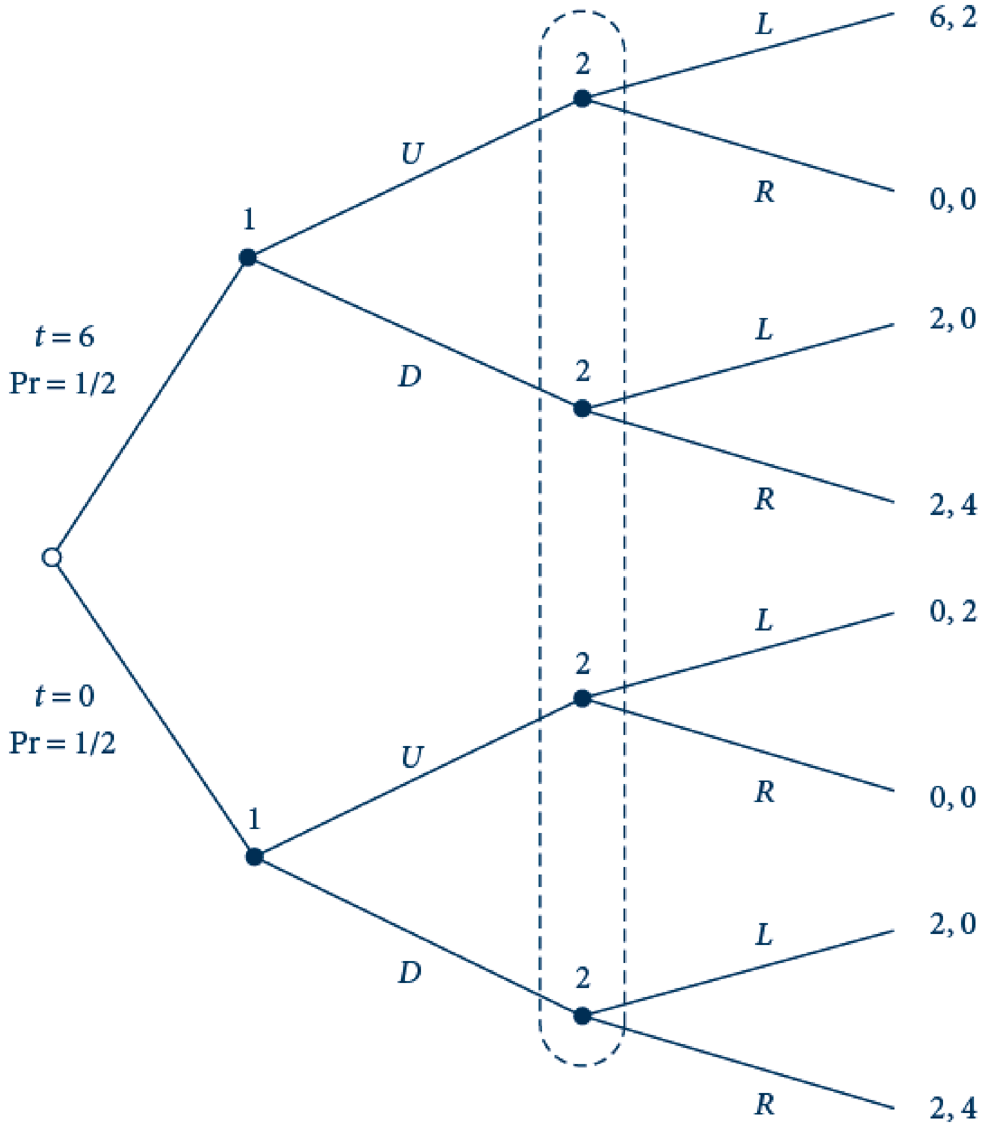
Simple Game of Incomplete Information

		Player 2	
		L	*R*
Player 1	*U*	t, 2	0, 0
	D	2, 0	2, 4

t = 6 with probability 1/2 and t = 0 with probability 1/2.

FIGURE 8.15

Extensive Form for Simple Game of Incomplete Information



This figure translates Figure 8.14 into an extensive-form game. The initial chance node is indicated by an open circle. Player 2's decision nodes are in the same information set because she does not observe player 1's type or action before moving.

Bayesian Nash equilibrium

- Bayesian Nash equilibrium: strategy profile (s_1,s_2,…s_n) where s_1=(s_1(t_{1a}), s_1(t_{1b}),…)

$$s_i(t_i) \in \arg\max_{s'_i \in S_i} \sum p(t_{-i} \mid t_i) u_i(s'_i, s_{-i}(t_{-i}), t)$$

- It means players are choosing best response given belief for each possible type

EXAMPLE 8.6 Bayesian–Nash Equilibrium of Game in Figure 8.15

- Two possible candidates for an equilibrium in pure strategies
  - 1 plays ($U|t$=6, $D|t$=0) and 2 plays L
    - Not an equilibrium
  - 1 plays ($D|t$=6, $D|t$=0) and 2 plays R
    - A Bayesian-Nash equilibrium

EXAMPLE 8.7 Tragedy of the Commons as a Bayesian Game

- Herder 1
  - Has private information regarding his value of grazing per sheep, $v_1(q_1,q_2,t)=t-(q_1+q_2)$
  - His type is
    - t=130 (the "high" type) with probability 2/3
    - t=100 (the "low" type) with probability 1/3
  - Value-maximization problem:

$$\max_{q_1}\{q_1 v_1(q_1,q_2,t)\}=\max_{q_1}\{q_1(t-q_1-q_2)\}$$

  - First-order condition: $t-2q_1-q_2=0$
  - So, $q_{1H}=65-q_2/2$ and $q_{1L}=50-q_2/2$

EXAMPLE 8.7 Tragedy of the Commons as a Bayesian Game

- Herder 2
  - Expected payoff:

$$\frac{2}{3}[q_2(120 - q_{1H} - q_2)] + \frac{1}{3}[q_2(120 - q_{1L} - q_2)] = q_2(120 - \bar{q}_1 - q_2)$$

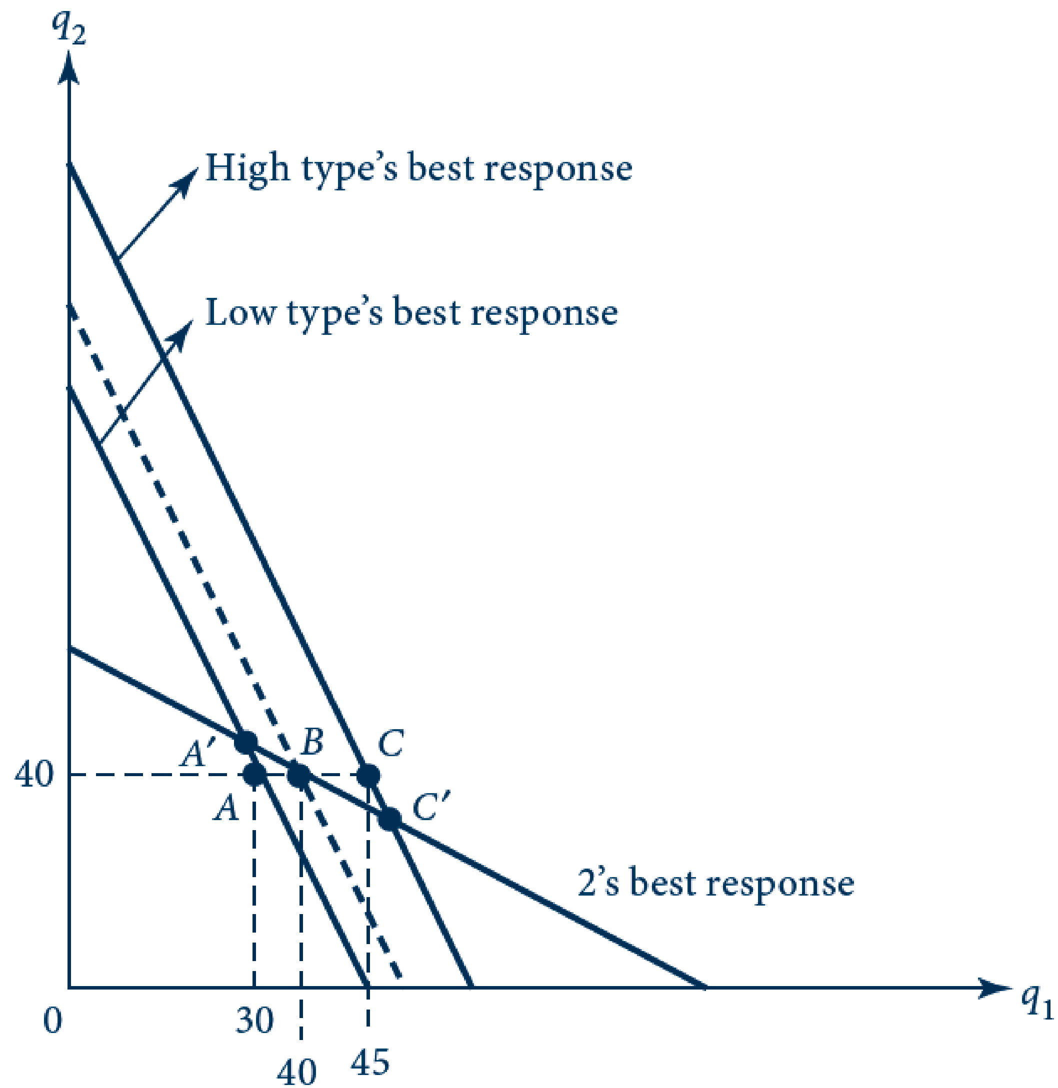
$$\text{where} \quad \bar{q}_1 = \frac{2}{3}q_{1H} + \frac{1}{3}q_{1L}$$

$$q_2 = 60 - \frac{\bar{q}_1}{2} = 30 + \frac{q_2}{4}$$

$$q_2^* = 40, q_{1H}^* = 45, q_{1L}^* = 30$$

FIGURE 8.16

Equilibrium of the Bayesian Tragedy of the Commons



Best responses for herder 2 and both types of herder 1 are drawn as thick solid lines; the expected best response as perceived by 2 is drawn as the thick dashed line. The Bayesian–Nash equilibrium of the incomplete-information game is given by points A and C; Nash equilibria of the corresponding full-information games are given by points A' and C' .

Dynamic Game of Incomplete Information

Perfect Bayesian Equilibrium

Signaling Game

Dynamic Games of Incomplete Information

- Multi-stage game with incomplete information
- **Perfect Bayesian Equilibrium**
  - Consists of a **strategy profile** and a **set of beliefs** such that at each information set
    - The strategy of a player moving there maximizes his expected payoff
      - The expectation is taken with respect to his beliefs
    - The beliefs of the player moving there are formed using Bayes' rule

Perfect Bayesian Equilibrium

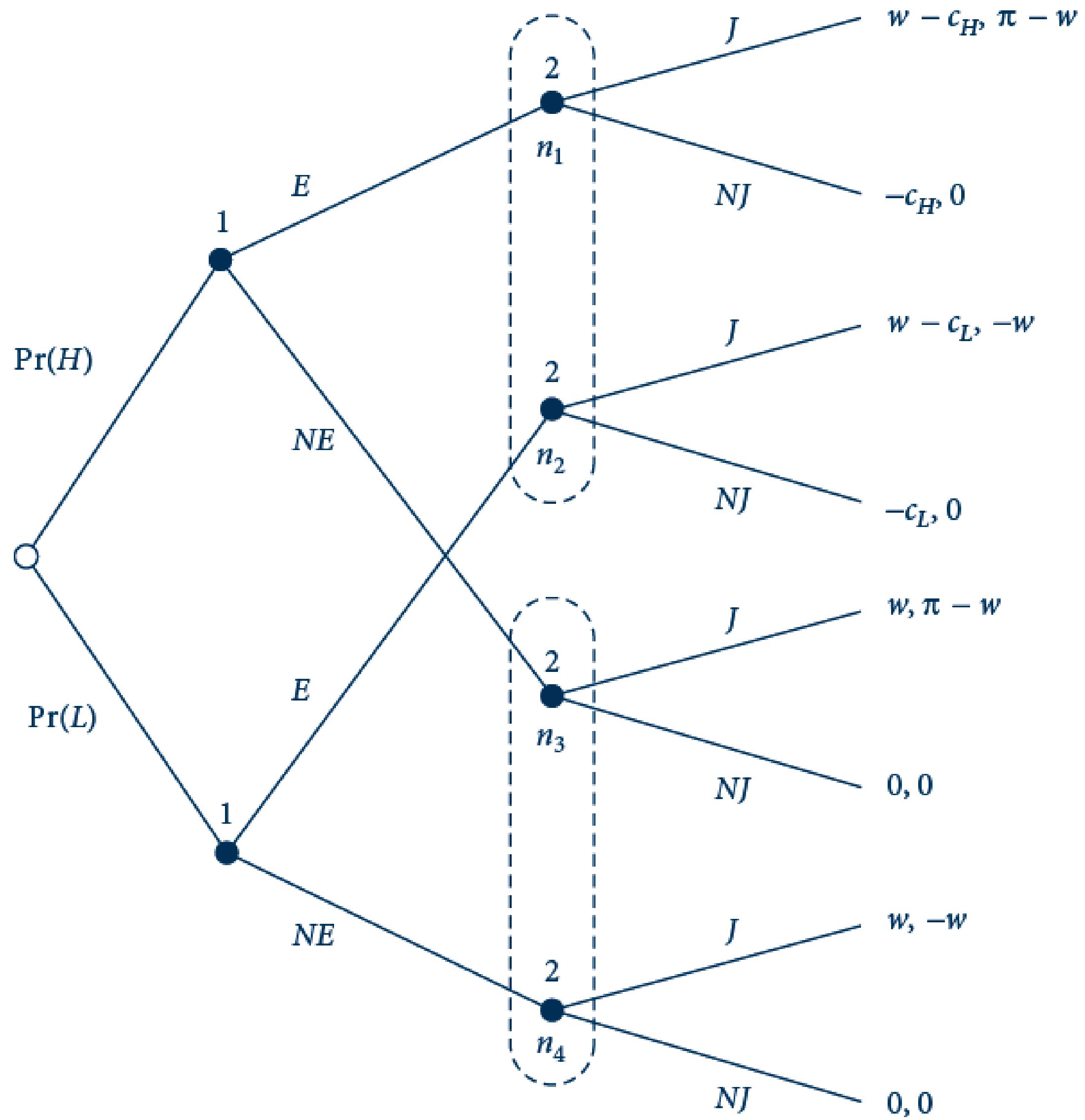
- A **strategy profile** and a **set of beliefs** such that at each information set
  - Player maximizes his expected payoff
    - The expectation is taken with respect to his beliefs
  - The beliefs of the player follows Bayes' rule

Signaling Game

- Two players sequential game with incomplete info.
- Informed player (player 1) learns own type and then takes an action
  - Type: Highly skilled (t = H)/ Low skilled (t = L)
  - Action: Education (s=H)/ No education (s=L)
  - c_L/c_H be the cost of obtaining an education for the low/high: Assume $c_H < c_L$
- Player 2 observes the action, update belief and moves
  - Action: Hire (wage $\pi > w > 0$)/No Hire (pay nothing)
  - Revenue: no revenue (t=L)/revenue of π (t=H)

FIGURE 8.17

Job-Market Signaling



Player 1 (worker) observes his or her own type. Then player 1 chooses to become educated (E) or not (NE). After observing player 1's action, player 2 (firm) decides to make him or her a job offer (J) or not (NJ). The nodes in player 2's information sets are labeled n_1,…, n_4 for reference.

Solving the game

- Player 2 (Backward induction)
  - Only observes player 1's action (education signal)
  - Expected payoff from playing *J* is

$$\Pr(H/E)(\pi - w) + \Pr(L/E)(-w) = \Pr(H/E)\pi - w$$

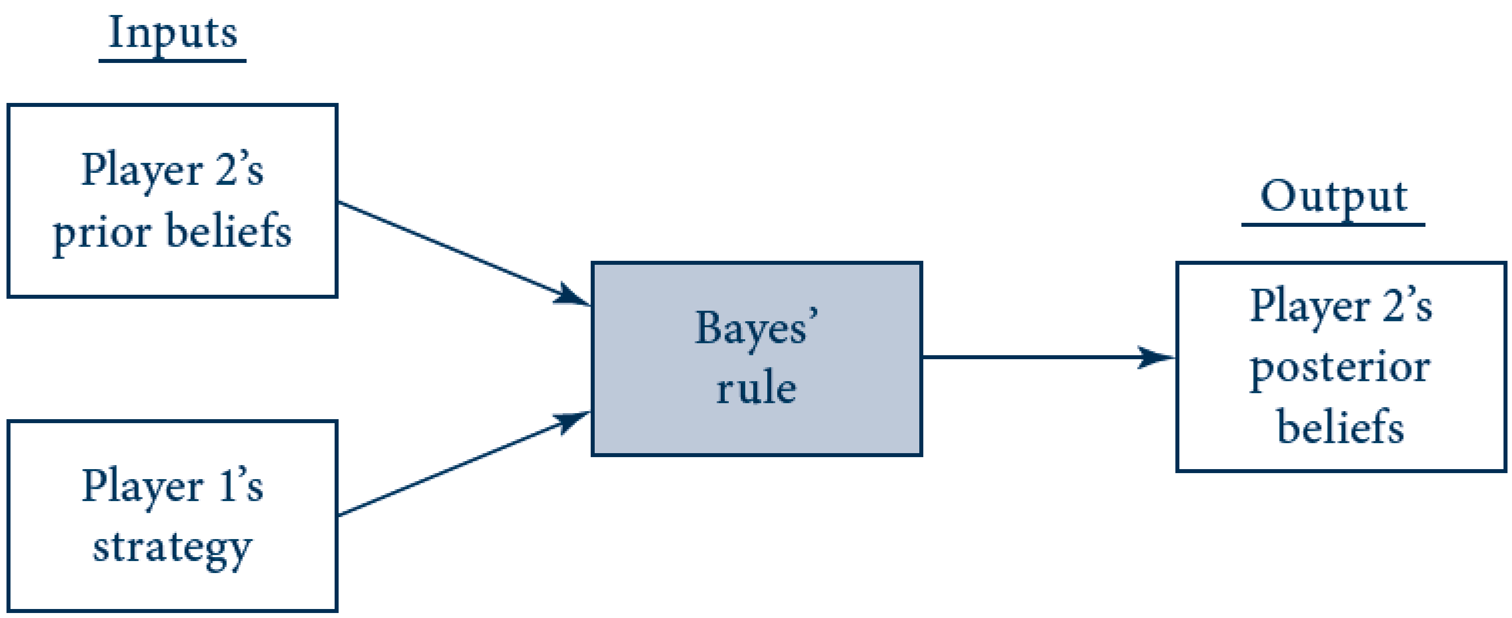
- By Bayes rule

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E|H)\Pr(H) + \Pr(E|L)\Pr(L)}$$

$$\Pr(H|NE) = \frac{\Pr(NE|H)\Pr(H)}{\Pr(NE|H)\Pr(H) + \Pr(NE|L)\Pr(L)}$$

FIGURE 8.18

Bayes' Rule as a Black Box



Bayes' rule is a formula for computing player 2's posterior beliefs from other pieces of information in the game.

Three Different PBEs

- Separating equilibrium
  - Each type of player chooses a different action
  - Player 2 learns player 1’s type with certainty after 1 moves
- Pooling equilibrium
  - Different types of player 1 choose the same action
  - Observing player 1’s move provides player 2 with no additional information
- Hybrid equilibrium
  - One type of player 1 plays a strictly mixed strategy
  - Player 2 learns a little about player 1’s type but doesn’t learn it with certainty
  - Player 2 may respond by also playing a mixed strategy

EXAMPLE 8.8 Separating Equilibrium in the Job-Market Signaling Game

- High-skilled worker
  - Signals his or her type by getting an education
- Player 2's beliefs
  - Pr(H|E)=Pr(L|NE)=1
  - Pr(H|NE)=Pr(L|E)=0
- Player 2's response
  - Offer a job if he observes that player 1 obtains an education
  - Not offer a job if he observes that player 1 does not obtain an education

- Check: player 1 would not want to deviate from the separating strategy (E|H, NE|L)
  - Given that player 2 plays (J|E, NJ|NE)
- Separating equilibrium
  - The worker obtains an education if and only if he is high-skilled
  - The firm offers a job only to applicants with an education if and only if $c_H < w < c_L$
- Another possible separating equilibrium
  - Player 1 to obtain an education if and only if he or she is low-skilled

EXAMPLE 8.9 Pooling Equilibria in the Job-Market Signaling Game

- To be a pooling equilibrium
  - In which both types of player 1 choose E
  - We need $Pr(H|NE) \leq w/\pi \leq Pr(H)$
  - Pr(H) must be sufficiently high
  - Pr(H|NE) must be sufficiently low
  - Type L pools with type H to prevent player 2 from learning anything about the worker’s skill from the education signal

EXAMPLE 8.10 Hybrid Equilibria in the Job-Market Signaling Game

- Hybrid equilibrium
  - Type H always to obtain an education
  - Type L to randomize
    - Between playing E and NE with probabilities e and 1 – e
  - Player 2's strategy
    - Offer a job to an educated applicant with probability j
    - Not to offer a job to an uneducated applicant

$$e^* = \frac{(\pi - w)\Pr(H)}{w\left[1 - \Pr(H)\right]}$$

Extensions

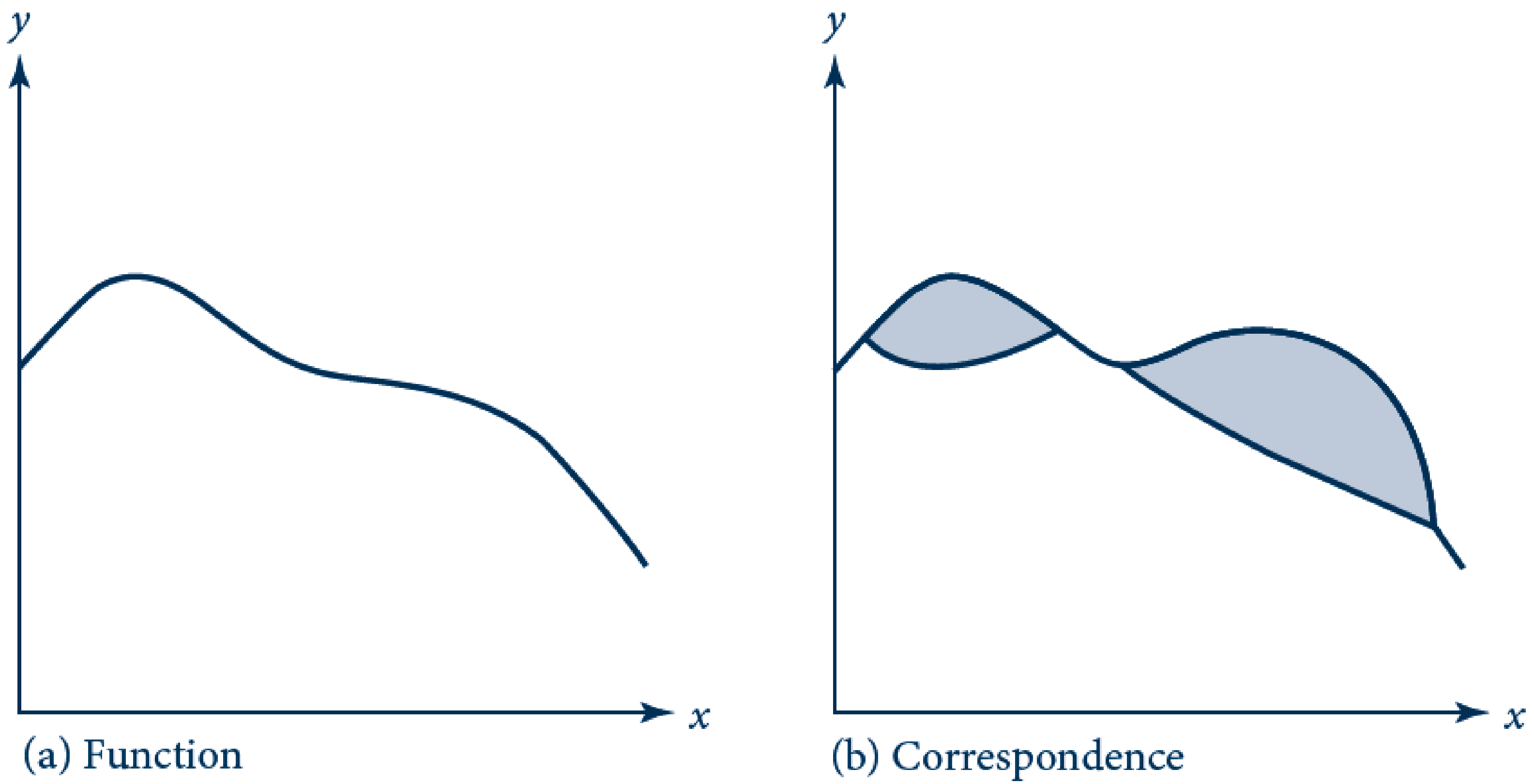
- Experimental economics
  - Explores how well economic theory matches the behavior or experimental subjects in a laboratory setting
- Evolutionary model
  - Players do not make rational decisions
    - They play the way they are genetically programmed
  - The more successful a player's strategy in the population
    - The more fit is the player
    - The more likely will the player survive to pass his or her genes on to future generations

Existence of Nash equilibrium

- A function
  - Maps each point in a first set to a single point in a second set
- A correspondence
  - Maps a single point in the first set to possibly many points in the second set
  - E.g.: the best response

FIGURE E 8.1

Comparison of Functions and Correspondences



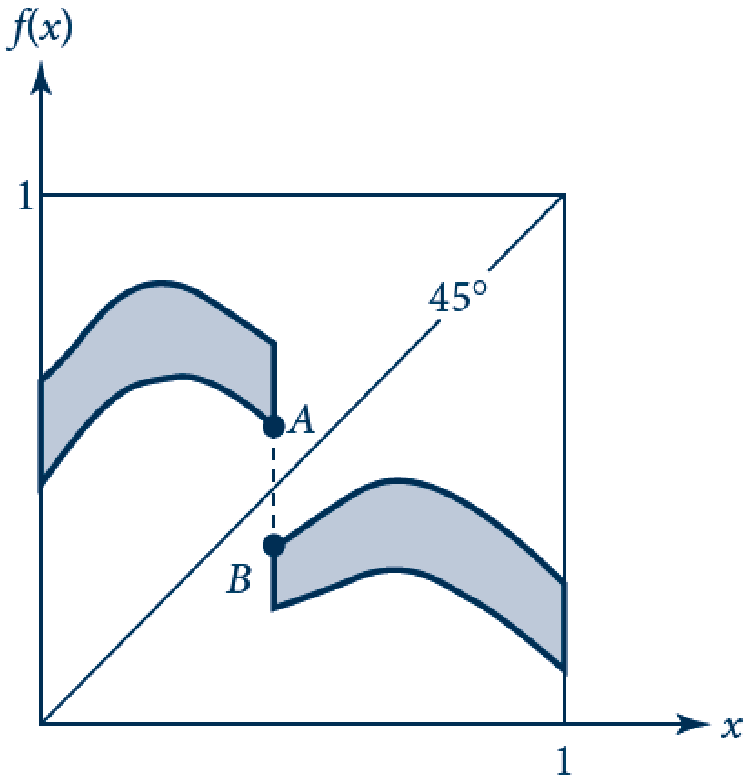
The function graphed in (a) looks like a familiar curve. Each value of x is mapped into a single value of y. With the correspondence graphed in (b), each value of x may be mapped into many values of y. Thus, correspondences can have bulges as shown by the shaded regions in (b).

Existence of Nash equilibrium

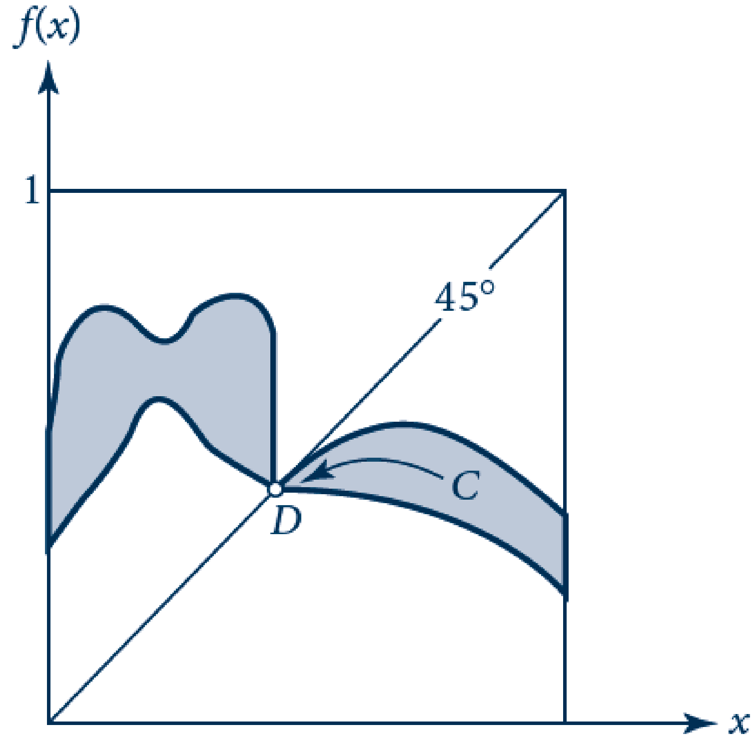
- Kakutani's fixed point theorem
  - Any convex, upper-semicontinuous correspondence [f(x)]
  - From a closed, bounded, convex set into itself
  - Has at least one fixed point (x^*) such that $x^* \in f(x)$

FIGURE E 8.2

Kakutani's Conditions on Correspondences



(a) Correspondence that is not convex



(b) Correspondence that is not upper semicontinuous

The correspondence in (a) is not convex because the dashed vertical segment between A and B is not inside the correspondence. The correspondence in (b) is not upper semi-continuous because there is a path (C) inside the correspondence leading to a point (D) that, as indicated by the open circle, is not inside the correspondence. Both (a) and (b) fail to have fixed points.

Existence of Nash equilibrium

- R(s)- the correspondence that underlies Nash's existence proof
  - Takes any profile of players' strategies $s = (s_1, s_2, \ldots, s_n)$
  - And maps it into another mixed strategy profile - the profile of best responses:
- $R(s) = (BR_1(s_{-1}); BR_2(s_{-2}), \ldots, BR_n(s_{-n}))$

EXTENSIONS

Existence of Nash equilibrium

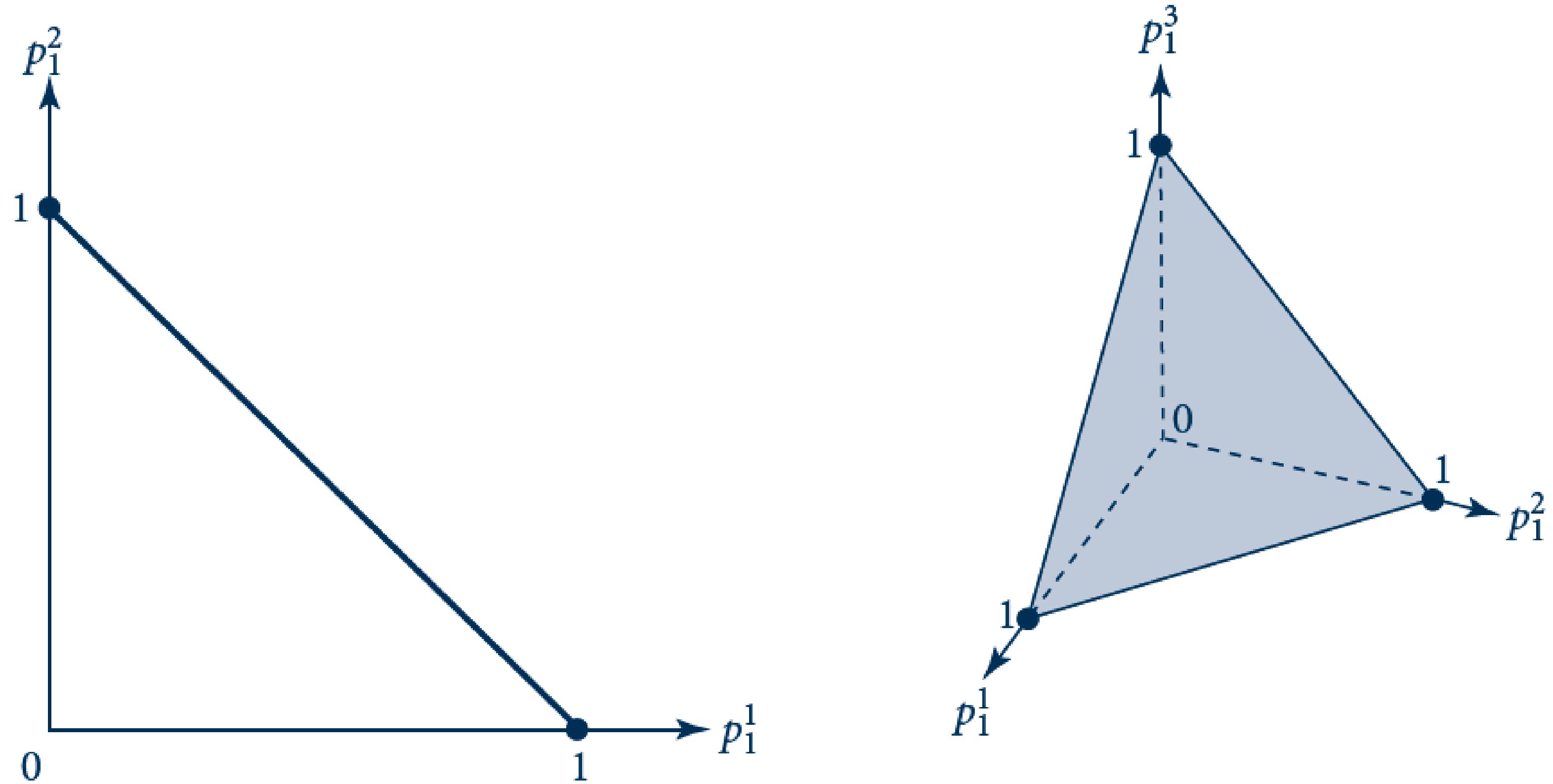
- Nash equilibrium
  - A fixed point of the correspondence is a strategy for which $s^* \in R(s^*)$
    - Each player's strategy is a best response to others' strategies
- The proof
  - Checks that all the conditions involved in Kakutani's fixed point theorem are satisfied by the best-response correspondence R(s)

Existence of Nash equilibrium

- The proof
  - Show that the set of mixed-strategy profiles is closed, bounded, and convex
  - Check that the best-response correspondence R(s) is convex
  - Check that R(s) is upper semicontinuous

FIGURE E 8.3

Set of Mixed Strategies for an Individual



(a) Two actions

(b) Three actions

Player 1’s set of possible mixed strategies over two actions is given by the diagonal line segment in (a). The set for three actions is given by the shaded triangle on the three-dimensional graph in (b).