EC4101
Topic 2: Theory of Firm

KO Chiu Yu

September 5, 2012

1 Reading

1. Snyder and Nicholoson, Chapter 9-11, Microeconomic Theory: Basic Principles and Extensions, 11th edition, 2012

2. Jehle and Reny, Chapter 3, Advanced Microeconomic Theory, 3rd edition, 2011

2 Production Functions

1. **Production function**: how production technology transforms inputs (factor of production) to output. Quantity q is related to factors r_1, r_2,..., r_n

$$q = f(r_1, r_2, \ldots, r_n)$$

In the usual capital-labor case, we have

$$q = f(k, l)$$

2. **Marginal product**: how many more output can be prodcued given extra unit of input.

$$\begin{aligned} MP_k &= \frac{\partial q}{\partial k} = f_k \\ MP_l &= \frac{\partial q}{\partial l} = f_l \end{aligned}$$

3. Diminishing marginal product: standard assumption

$$\begin{aligned} \frac{\partial MP_k}{\partial k} &< 0 \Leftrightarrow f_{kk} < 0 \\ \frac{\partial MP_l}{\partial l} &< 0 \Leftrightarrow f_{ll} < 0 \end{aligned}$$

4. **Average product: (**Note that average product is calculated holding other factor constant; so it is depend on the level of other factors)

$$AP_k = \frac{f(k,l)}{k}, AP_l = \frac{f(k,l)}{l}$$

5. **Isoquant**:

   (a) iso is Greek word meaning "equal"; so isoquant = equal quantity. Set of different input combinations producing the same output

   (b) graphical representation of production function (recall: indifference curve is graphcial representation for preference)

6. Marginal rate of Technical Substitution: change in one input

   (a) Marginal rate of Technical Substitution = - Slope of Isoquant (recall: slope of the tangent on a point of indifference curve is the marginal rate of substitution)

   $$MRTS_{l,k} = -\left.\frac{dk}{dl}\right|_{q=q_0} = \frac{MP_l}{MP_k} = \frac{f_l}{f_k}$$

   (b) MRTS is always positive (why?)

   (c) Diminishing return does not implies diminishing MRTS (similar to diminishing marginal does not implies diminishing MRS)

   (d) Convex Isoquant is equivalent diminishing MRTS

   $$\frac{dMRTS_{l,k}}{dl} < 0$$

7. **Return to Scale**: change in all inputs

(a) How outputs change if you have multiple of all inputs

$$\begin{aligned} f(tk,tl) &> tf(k,l) \Rightarrow \text{increasing return to scale} \\ f(tk,tl) &= tf(k,l) \Rightarrow \text{constant return to scale} \\ f(tk,tl) &< tf(k,l) \Rightarrow \text{decreasing return to scale} \end{aligned}$$

(b) Constant return to scale:

i. Homogeneous of degree one in inputs (why?)

ii. Marginal product functions is homogenous of degree zero in inputs (Math results)

iii. Isoquants are radial expansions (why?)

8. **Elasticity of Substitution**:

(a) it measures survature of Isoquant

(b) how proportionate change in k/l relative change to RTS

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta RTS} = \frac{d\ln(k/l)}{d\ln(f_l/f_k)}$$

(c) Graphically, if you rotate about the origin, how will the slope of Isoquant changes. Hence,

i. σ is high: isoquant is flat

ii. σ is low: isoquant is curved

9. Four standard production function

(a) Linear Production: $q = \alpha k + \beta l$ where $\alpha, \beta > 0$ (c.f. perfect substitute!)

(b) Fixed Proportion: $q = \min\{\alpha k, \beta l\}$ where $\alpha, \beta > 0$ (c.f. perfect complement!)

(c) Cobb-Douglas: $q = Ak^\alpha l^\beta$ where $\alpha, \beta > 0$ (c.f. Cobb-Douglas?!)

(d) Constant Elasticity of Substitution (CES): $q = (k^\rho + l^\rho)^{\gamma/\rho}$ where $\rho \leq 1$, $\rho \neq 0$ and $\gamma > 0$ (c.f. CES?!)

3 Cost Functions

1. **Cost minimization problem**: given production target q_0, how to produce it with lowest cost by varying inputs

$$\min_{l,k} wl + vk \text{ such that } q_0 = f(k,l)$$

Then the Lagrangian would be

$$\mathcal{L} = wl + vk + \lambda \left[q_0 - f(k,l) \right]$$

so that FOCs are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial l} &= w - \lambda f_l = 0 \\ \frac{\partial \mathcal{L}}{\partial k} &= v - \lambda f_k = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= q_0 - f(k,l) = 0 \end{aligned}$$

Three interpretation of the optimal conditions:

(a) Price ratio (external exchange rate) = Marginal rate of Technical substitution (internal exchange rate)

$$\frac{w}{v} = \frac{f_l}{f_k} = RTS_{l,k}$$

(b) Marginal productivity per dollar equalized across inputs

$$\frac{f_k}{v} = \frac{f_l}{w}$$

(c) Extra cost for extra input equalized

$$\frac{w}{f_l} = \frac{v}{f_k} = \lambda$$

2. **Contingent Demand (or Conditional input Demand or Derived Demand)** for inputs:

(a) Optimal solution to this minimization problem

$$
\begin{aligned}
l^c &= l(w, v, q) \\
k^c &= k(w, v, q)
\end{aligned}
$$

(b) Can be derived from Shepherd's lemma

$$
\begin{aligned}
\frac{\partial C}{\partial w} &= l^c \\
\frac{\partial C}{\partial v} &= k^c
\end{aligned}
$$

3. **Firm's expansion path**:

   (a) path of contingent demands l^c and k^con increasing q.

   (b) Note that this is not monotonic. If it is decreasing, it is inferior input. (Recall: copying technology: rubber band and copy paper)

4. **Total cost function**:

   (a) Optimal value function to the cost minimization problem

$$
C(v, w, q) = \; wl^c(w, v, q) + vk^c(w, v, q)
$$

   (b) Properties:

      i. Homoegenous of degree one in input prices (why?)

      ii. Nondecreasing in q, v and w (why?)

      iii. Concave in input prices (why?)

   (c) **Average cost**: $AC(v, w, q) = \dfrac{C(v, w, q)}{q}$

   (d) **Marginal cost**:$MC(v, w, q) = \dfrac{\partial C(v, w, q)}{\partial q}$

5. Long run versus Short run

   (a) **Short-run**: some factors are fixed

   (b) **Long-run**: all factors are free to vary

(c) **Short-run total cost:** wsually, capital is assumed to be fixed at k_1 in the short-run

$$SC(w, k_1, q) = vk_1 + wl^c(w, k_1, q)$$

Note that $l^c(w, k_1, q)$ is obtained from cost minization problem holding capital at k_1.

(d) **Short-run average total cost**: $SAC(w, k_1, q) = \dfrac{SC(w, k_1, q)}{q}$

(e) **Short-run marginal cost**: $SMC(w, k_1, q) = \dfrac{\partial SC(w, k_1, q)}{\partial q}$

(f) MC curve crosses AC curve at minimium AC level (why? prove it)

(g) SAC curve is tangent to the AC curve (envelop theorem)

(h) SMC intersects MC at the output that SAC curve is tangent to the AC curve (why?)

(i) Using the above, we have $AC = MC = SAC = SMC$ at minimum point of AC curve.

4 Profit-Maximization

1. **Profit maximization problem**: maximize profit by varying production level

$$\max_q \pi = p(q)q - c(q)$$

where $p(q)$ is the inverse demand function and $c(q)$ is the total cost function from the previous section (the factor prices are subsumed).

FOC is

$$\frac{d\pi}{dq} = p(q) + q\frac{dp}{dq} - \frac{dc}{dq} = 0$$

or marginal revenue = marginal cost

$$MR(q) = p(q) + q\frac{dp}{dq} = MC(q)$$

SOC

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} < 0$$

2. **Price elasticity of demand:** responsiveness of quantity demanded respective to change in price

$$e_{q,p} = \frac{dq}{dp}\frac{p}{q}$$

   (a) Marginal revenue: $MR(q) = p(q) + q\frac{dp}{dq} = p(q)\left(1 + \frac{1}{e_{q,p}}\right)$

   (b) Infinitely elastic demand: $e_{q,p} = -\infty$

   (c) Elastic demand: $e_{q,p} < -1$

   (d) Unit Elastic demand: $e_{q,p} = -1$

   (e) Inelastic demand: $e_{q,p} < -1$

   (f) Inverse elasticity rule: higher elasticity, lower the markup

$$\text{Markup} = \frac{p - MC}{p} = -\frac{1}{e_{q,p}} = \frac{1}{|e_{q,p}|}$$

3. Short-run supply curve of a price-taking firm:

   (a) Facing horizontal demand curve: $P = MR$

   (b) Operate only if $P > SAVC$; Otherwise, shutdown

   (c) Hence, supply curve is SMC above $SAVC$ curve

4. **Profit maximization problem**: maximize profit by varying inputs

$$\max_{k,l} \pi = pf(k,l) - vk - wl$$

   FOCs are

$$\begin{aligned} \frac{d\pi}{dk} &= pf_k - v = 0 \\ \frac{d\pi}{dl} &= pf_l - w = 0 \end{aligned}$$

   Two implications:

   (a) Cost minimization

$$RTS_{l,k} = \frac{w}{v}$$

(b) **marginal revenue product** (marginal contribution) = input price (marginal cost)

$$\begin{aligned} pf_k &= v \\ pf_l &= w \end{aligned}$$

SOCs are

$$\begin{aligned} \pi_{kk} &= f_{kk} < 0 \\ \pi_{ll} &= f_{ll} < 0 \\ \pi_{kk}\pi_{ll} - \pi_{kl}^2 &= f_{kk}f_{ll} - f_{kl}^2 > 0 \end{aligned}$$

5. (**Unconditional) input Demand or Derived Demand)** for inputs: demand for inputs only depends on input and output prices

   (a) Optimal solution to profit maximization problem

$$\begin{aligned} l^* &= l(p, w, v) \\ k^* &= k(p, w, v) \end{aligned}$$

   (b) Can be derived from Envelop theorem

$$\begin{aligned} \frac{\partial \Pi}{\partial w} &= -l^* \\ \frac{\partial \Pi}{\partial k} &= -k^* \end{aligned}$$

   (c) Single-input: input demand is downward sloping

   FOC is $Pf_l - p = 0$. Let $F(l, w, p) = Pf_l - p$Then

$$\frac{dl}{dw} = -\frac{\partial F/\partial w}{\partial F/\partial l} = \frac{1}{pf_{ll}} \leq 0$$

   (d) Two-input case:

      i. Own price effect is always negative (no Giffen paradox!)

$$l(p, v, w) = l^c(v, w, q) = l^c(v, w, q(p, v, w))$$

Then
$$\frac{\partial l(p,v,w)}{\partial w} = \underbrace{\frac{\partial l^c(v,w,q)}{\partial w}}_{\textbf{substituion effect}} + \underbrace{\frac{\partial l^c(v,w,q)}{\partial q}\frac{\partial q}{\partial w}}_{\textbf{output effect}} < 0$$

Substitution effect is obviously negative but the output effect is negative will be explained in the Appendix.

ii. Cross price effect is mixed.

6. **Profit function**

(a) optimal-value function of profit maximization problem

$$\Pi(w,v,P) = pf(k(P,w,v), l(P,w,v)) - vk(P,w,v) - wl(P,w,v)$$

(b) Properties

i. Homogeneous of degree one in all prices (why?)

ii. Nondecreasing in output price p (why?)

iii. Nonincreasing in input prices v and w

iv. Convex in output prices

$$\frac{\Pi(p_1,v,w) + \Pi(p_2,v,w)}{2} \geq \Pi(\frac{p_1+p_2}{2}, v, w)$$

(c) Envelop theorem implies
$$q(P,v,w) = \frac{\partial \Pi}{\partial p}$$

7. Producer Surplus:

$$\begin{aligned} PS &= \Pi(p^*,\ldots) - \Pi(p_s,\ldots) \\ &= \int_{p_s}^{p^*} q(p)dp \text{ [by Envelop Theorem]} \end{aligned}$$

where p_s is the shutdown price

5 Appendix.

1. Negative output effect

$$
\begin{aligned}
\frac{\partial l^c(v,w,q)}{\partial q}\frac{\partial q}{\partial w} &= \frac{\partial l^c(v,w,q)}{\partial q}\frac{\partial}{\partial w}\frac{\partial \Pi}{\partial p} \\
&= \frac{\partial l^c(v,w,q)}{\partial q}\frac{\partial}{\partial p}\frac{\partial \Pi}{\partial w} \text{ [Young's Theorem]} \\
&= \frac{\partial l^c(v,w,q)}{\partial q}\left(-\frac{\partial l}{\partial p}\right) \text{ [Envelop Theorem]} \\
&= -\left[\frac{\partial l^c(v,w,q)}{\partial q}\right]^2 \frac{\partial q}{\partial p} \; [l(p,v,w) = l^c(v,w,q(p,v,w))] \\
&= -\left[\frac{\partial l^c(v,w,q)}{\partial q}\right]^2 \frac{\partial}{\partial p}\frac{\partial \Pi}{\partial p} \text{ [Envelop Theorem]} \\
&= -\left[\frac{\partial l^c(v,w,q)}{\partial q}\right]^2 \frac{\partial^2 \Pi}{\partial p^2} \\
&< 0 \text{ [}\Pi\text{ is convex]}
\end{aligned}
$$