EC4101
Microeconomics Analysis III
(Group 2)

Topic 4

Uncertainty

Basic Concept of Probability Theory

- Sample Space
- Outcomes
- Experiment/Random Trail

Basic Concept of Probability Theory

- Random Variable:
  - not a variable that is random
  - function from outcome space to numbers
- Discrete and Continuous
- Cumulative distribution function (cdf)
- Probability mass function (pmf)
- Probability distribution function (pdf)

Basic Concept of Probability Theory

- Expected value: E(x)

$$E(x) = \sum_{i=1}^{n} x_i f(x_i) \qquad E(x) = \int_{-\infty}^{+\infty} x f(x) dx$$

- Variance

$$\text{Discrete: } Var(x) = \sigma_x^2 = \sum_{i=1}^{n} \left[x_i - E(x)\right]^2 f(x_i)$$

$$\text{Continuous: } Var(x) = \sigma_x^2 = \int_{-\infty}^{+\infty} \left[x - E(x)\right]^2 f(x) dx$$

Lottery

- Two-outcome lottery: $(x,y;p,1-p)$
- Win x with probability p
- Win y with probability $1-p$
- Expected value of lottery $h= (x,y;p,1-p)$

$$E(h)=px+(1-p)y$$

Fair Bet

- Lottery h is a fair bet if expected value of zero:

$$E(h)=0$$

- Utility (lottery) = E(lottery)?

St. Petersburg Paradox:

- A coin is flipped until a head appears
- If a head appears on the nth flip, the player is paid $2^n
- EX = ∞

$$E(x) = \sum_{i=1}^{\infty} \pi_i x_i = \sum_{i=1}^{\infty} 2^i \left(\frac{1}{2}\right)^i = 1+1+1+\ldots+1+\ldots = \infty$$

What's wrong here?

- Not willing to pay a large amount for infinite expected value lottery
- U(lottery)≠E(lottery)
- So we need something more…

Expected Utility

- von Neumann-Morgenstern Theorem
  - Lottery can be ranked by expected utility

$$EU(h) = \sum_{i=1}^{n} \pi_i U(x_i)$$

  - Cardinal utility
  - Expected utility maximization
    - individuals act as if they are maximizing EU
- St. Petersburg game may converge to a finite expected utility value

EXAMPLE 7.1 Bernoulli's Solution to the Paradox and Its Shortcomings

- Utility of each prize in the St. Petersburg paradox is $U(x_i)=ln\ x_i$
  - Diminishing marginal utility (U' > 0 but U'' < 0),
  - The expected utility value of this game converges to a finite number:

$$\text{expected utility} = \sum_{i=1}^{\infty} \pi_i U(x_i) = \sum_{i=1}^{\infty} \frac{1}{2^i} \ln(2^i) = 1.39$$

EXAMPLE 7.1 Bernoulli's Solution to the Paradox and Its Shortcomings

- Bernoulli's solution to the St. Petersburg paradox
  - Does not completely solve the problem
  - As long as there is no upper bound to the utility function
    - The paradox can be regenerated by redefining the gamble's prizes

Assumptions for vNM Utility

- Completeness:
  - for any lottery x,y; either xRy; yRx or both
- Transitivity:
  - If xRy and yRz, then xRz
- Continuity:
  - If xRyRz, then there exists $0 \leq p \leq 1$ such that $px+(1-p)z$ is indifferent with y
- Independence:
  - If xRy, then for any lottery z, and $0 \leq p \leq 1$, we have $px+(1-p)z \; R \; py+(1-p)z$

(Absolute) Certainty Equivalent

- (Absolute) Risk Premium (RP): amount needed to take lottery h given wealth W

$$U(W\text{-}RP) = EU(h)$$

- (Absolute) Certainty equivalent (CE) of lottery h=(x,y;p,1-p) is

$$U(CE)=EU(h)=pU(x)+(1-p)U(y)$$

- By construction, U(CE) = U(W-RP), so

$$CE= W\text{-}RP$$

(Absolute) Risk Premium

- (Absolute) Risk Premium (RP): amount needed to take lottery h given wealth W

$$U(W+E(h)-RP) = EU(W+h)$$

- (Absolute) Certainty equivalent (CE) of lottery h=(x,y;p,1-p) is

$$U(CE)=EU(W+h)=pU(W+x)+(1-p)U(W+y)$$

- By construction, U(CE) = U(W+E(h)-RP), so

$$CE= W+E(h)-RP$$

Relative Risk Premium

- Relative Risk Premium (RRP): relative amount of wealth needed to take lottery hW given wealth W

$$U(E(Wh)\text{-}W\times RRP) = EU(hW)$$

- By rewriting, we have

$$W\times RRP(h,W) = RP(hW,W)$$

Risk Attitude

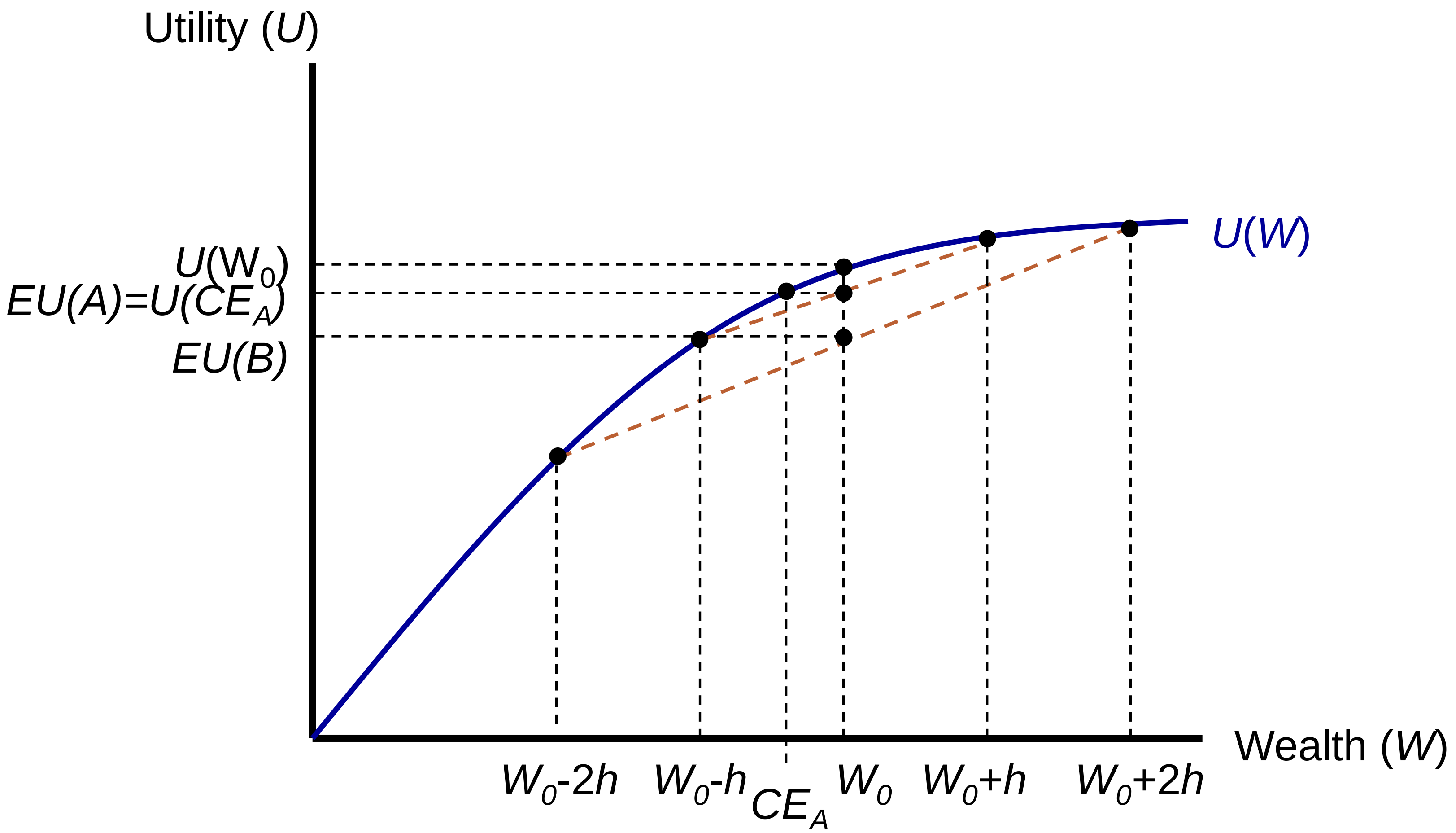
- Risk attitude: sign of risk premium for fair bet:
  - Risk loving: risk premium <0
  - Risk neutral: risk premium =0
  - Risk aversed: risk premium > 0
- St. Petersburg paradox: most of us are risk-aversed
- Size of Risk premium: most natural measure

Risk Attitude

- Risk loving: risk premium <0
- Risk neutral: risk premium =0
- Risk aversed: risk premium > 0

FIGURE 7.1

Utility of Wealth from Two Fair Bets of Differing Variability



If the utility-of-wealth function is concave (i.e., exhibits a diminishing marginal utility of wealth), then this person will refuse fair bets. A 50–50 chance of winning or losing h dollars, for example, yields less expected utility [EU(A)] than does refusing the bet. The reason for this is that winning h dollars means less to this individual than does losing h dollars.

EXAMPLE 7.2 Willingness to Pay for Insurance

- A person with a current wealth of $100,000
  - Faces a 25% chance of losing his automobile worth $20,000
  - Von Neumann-Morgenstern utility index is:

    $U(W) = \ln(W)$
  - Expected utility without insurance
    - EU(no insurance) = 0.75U(100,000) + 0.25U(80,000) = 0.75 ln100,000+0.25 ln80,000 = 11.45714
  - Expected utility with insurance
    - EU(fair insurance)=U(95,000)=ln 95,000 = 11.46163

EXAMPLE 7.2 Willingness to Pay for Insurance

- EU(maximum-premium insurance)= U(100,000 – x) = ln (100,000 – x) =11.45714
- So x=5,426

- This person
  - Would be willing to pay up to $426 in administrative costs to an insurance company
  - In addition to the $5,000 premium to cover the expected value of the loss
  - Is as well off as he or she would be when facing the world uninsured

Risk Aversion

- Refuse fair bet (Prefer EX over X)
- willing to pay something to avoid taking fair bets (risk premium)
- explain why insurance (unfair bet)
- Intuition: marginal utility of wealth falls as wealth gets larger

Measure Risk Aversion

- Most direct measure: risk premium
  - Absolute
  - Relative
- Both are cumbersome to calculate
- Many measures: classic one is by Arrow-Pratt

Risk Aversion Measure

- Absolution size: **absolute risk aversion (ARA)**

$$r(W) = -\frac{U''(W)}{U'(W)}$$

- Relative size: **relative risk aversion (RRA)**

$$rr(W) = Wr(W) = -W\frac{U''(W)}{U'(W)}$$

Risk Aversion and Wealth

- If utility is quadratic in wealth,

$U(W) = a + bW + cW^2$

  - Where $b > 0$ and $c < 0$
  - Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U(W)} = \frac{-2c}{b + 2cW}$$

    - Risk aversion increases as wealth increases

Risk Aversion and Wealth

- If utility is logarithmic in wealth,

  $U(W) = ln\ (W\ )$

  – Where $W > 0$

  – Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U(W)} = \frac{1}{W}$$

    - Risk aversion decreases as wealth increases

Risk Aversion and Wealth

- If utility is exponential,

$U(W) = -e^{-AW} = -exp\,(-AW)$

  - Where A is a positive constant
  - Pratt's risk aversion measure is

$$r(W) = -\frac{U''(W)}{U(W)} = \frac{A^2 e^{-AW}}{Ae^{-AW}} = A$$

    - Risk aversion is constant as wealth increases

Constant Risk Aversion

- How much (*f*) would an individual pay to avoid the risk?
  - Initial wealth is W_0
  - Utility function exhibits constant absolute risk aversion
  - A 50–50 chance of winning or losing $1,000
  - To find *f*, we set the utility of $W_0 - f$ equal to the expected utility from the gamble

$$-\exp[-A(W_0-f)] = -0.5\exp[-A(W_0+1{,}000)] - 0.5\exp[-A(W_0-1{,}000)]$$

$$\exp(Af) = 0.5\exp(-1{,}000A) + 0.5\exp(1{,}000A)$$

Relative Risk Aversion

- The power utility function

$$\mathrm{U(W,\ R)} = \begin{cases} W^R / R \quad \text{for} \quad R < 1, R \neq 0 \\ \ln R \ \ \text{if } R = 0 \end{cases}$$

  – Diminishing absolute relative risk aversion

$$r(W) = -\frac{U''(W)}{U'(W)} = -\frac{(R-1)W^{R-2}}{W^{R-1}} = -\frac{(R-1)}{W}$$

  – But constant relative risk aversion

$$rr(W) = Wr(W) = -(R-1) = 1 - R$$

- Constant relative risk aversion utility function
  - What fraction of initial wealth (f)
    - Willing to give up to avoid a fair gamble of, 10% of initial wealth
  - Assume R = 0
  - Logarithmic utility function

    $ln[(1-f)W_0] = 0.5\ ln(1.1W_0) + 0.5\ ln(0.9W_0)$

    $ln(1-f) = 0.5\ ln(1.1) + 0.5\ ln(0.9) = ln(0.99)^{0.5}$

    $f=0.005$
  - Sacrifice up to 0.5 percent of wealth to avoid the 10 percent gamble

Technical: Absolute Risk Aversion

- Consider a fair bet h (*$E(h) = 0$*)
- Absolute Risk premium (ARP(h;W)):

  $E[U(W + h)] = U(W – ARP(h;W))$
- Taylor series expansion:
- *LHS:* $U(W - ARP) = U(W) - ARP \times U'(W) + \ldots$
- *RHS:*

  $E[U(W + h)] = E[U(W) - hU'(W) + h^2/2\ U''(W) + \ldots$

  $E[U(W + h)] = U(W) - E(h)U'(W) + E(h^2)/2\ U''(W) + \ldots$

  $E[U(W + h)] = U(W) + Var(h)/2\ U''(W) + \ldots$

Technical: Absolute Risk Aversion

- Dropping higher order terms, we have

$$U(W) - ARP \times U'(W) \cong U(W) + \frac{1}{2} Var(h) U''(W)$$

$$ARP(h;W) \cong \frac{1}{2} Var(h) r(W)$$

Technical: Relative Risk Aversion

- Relative Risk premium (RRP):

$$E[U(W + h)] = U(W - RPR \times W)$$

- Hence, ARP(Wh;W) = W×RRP(h;W)

$$RRP(h;W) = \frac{ARP(Wh;W)}{W}$$

$$\approx \frac{1}{W}\frac{1}{2}Var(Wh)r(W) = \frac{1}{2}Var(h)Wr(W)$$

$$= \frac{1}{2}Var(h)rr(W)$$

Mean-Variance Preference

- ARP and RRP is approximately related to variance of lottery
- Is there any case that preference exactly represented by mean and variance of lottery?
  - Quadratic Utility ($U(w)=aw-bw^2$)
  - Constant absolute aversion and lottery follows normal distribution ($U(w)=-e^{-rW}$)

Quadratic utility

- Utility: $U(W)=aW-bW^2$
- Expected utility of lottery h:

  $EU(W+h)$

  $=aE(W+h)-bE((W+h)^2)$

  $= a(W+E(h))-b[Var(h)+(W+E(h))^2]$
- Hence, only E(h) and Var(h)

CARA with normality

- Lottery h follows normal distribution:

$$f(h) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(h-\mu)^2}{2\sigma^2} \right)$$

- CARA utility: U(X)=-exp(-rX)

E[U(W+h)]

$$= \int_{-\infty}^{\infty} U(W+h) f(h) dh$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int -\exp(rW + rh) \exp(-\frac{(h-\mu)^2}{2\sigma^2}) dh$$

CARA with Normality

- Con't from previous page

E[U(W+h)]

$$= \frac{-\exp(rW)}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{(h-\mu+r\sigma^2)^2 + r^2\sigma^4 - 2\mu r\sigma^2}{2\sigma^2}\right) dh$$

$$= \frac{-\exp(rW - \mu r + \frac{1}{2}r^2\sigma^2)}{\sqrt{2\pi\sigma^2}} \int \exp\left(-\frac{(h-\mu+r\sigma^2)^2}{2\sigma^2}\right) dh$$

$$= -\exp(rW - \mu r + \frac{1}{2}r^2\sigma^2)$$

Applications

- Diversification
- Pricing contingent commodities
- Insurance

Diversification

- A person has wealth W to invest in two independent risky assets, 1 and 2
  - Equal expected values ($\mu_1=\mu_2$)
  - Equal variances ($\sigma^2{}_1=\sigma^2{}_2$)
- Undiversified portfolio: just one of the assets
  – Expected return: $\mu_{UP} = \mu_1=\mu_2$
  – Variance: $\sigma^2{}_{UP} = \sigma^2{}_1=\sigma^2{}_2$

Diversification

- Diversified portfolio, DP
  - α_1 – the fraction invested in the first asset
  - $(1- \alpha_1)$ – the fraction invested in the second
  - Expected return:

$$\mu_{DP} = \alpha_1 \mu_1 + (1- \alpha_1)\mu_2 = \mu_1 = \mu_2$$

  - Variance:

$$\sigma^2{}_{DP} = \alpha^2{}_1 \sigma^2{}_1 + (1- \alpha_1)^2 \sigma^2{}_2 = (1-2\alpha_1 + 2\alpha^2{}_1)\, \sigma^2{}_1$$

  - Minimize $\sigma^2{}_{DP}$

$$\alpha_1 = ½\ ; \ \sigma^2{}_{DP} = \sigma^2{}_1/2$$

Contingent commodities

- Contingent commodities (e.g. insurance)
  - delivered only in a particular state of the world
  - "\$1 in good times" or "\$1 in bad times"
- Assume two contingent goods
  - Wealth in good/bad times (w_g/w_b)
  - probability that good times: π
  - Expected utility: $V(W_g,W_b) = \pi U(W_g) + (1 - \pi)U(W_b)$
- Budget constraint: $W = p_g W_g + p_b W_b$

Contingent commodities

- price ratio p_g / p_b
- If there is a market, all old techniques apply!
  - Fair price : $p_g = \pi$ and $p_b = (1 - \pi)$
  - Fair market:

    Price ratio = odds in favor of good times

$$\frac{p_g}{p_b} = \frac{\pi}{1-\pi}$$

- Optimality: $MRS = \dfrac{\partial V / \partial W_g}{\partial V / \partial W_b} = \dfrac{\pi U'(W_g)}{(1-\pi)U'(W_b)} = \dfrac{p_g}{p_b}$

Fair Market

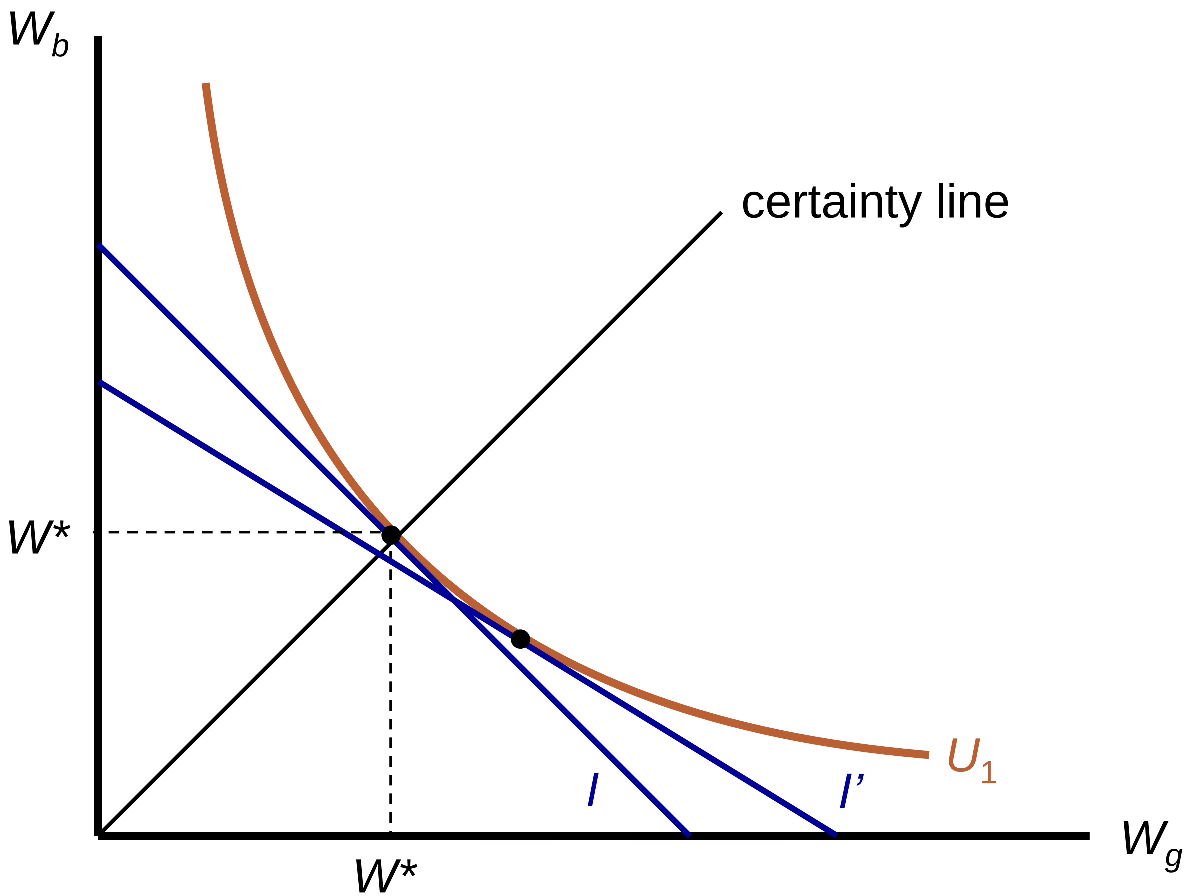
- Fair market:

$$\frac{U'(W_g)}{U'(W_b)} = 1 \quad \text{or} \quad W_g = W_b$$

- Individual makes the same level of wealth regardless of the state (full coverage!)

FIGURE 7.5

Risk Aversions in the State-Preference Model



The line I represents the individual's budget constraint for contingent wealth claims: $W = p_g W_g + p_b W_b$. If the market for contingent claims is actuarially fair $[p_g / p_b = \pi/(1- \pi)]$, then utility maximization will occur on the certainty line where $W_g = W_b = W^*$. If prices are not actuarially fair, the budget constraint may resemble I', and utility maximization will occur at a point where $W_g > W_b$.

- A person with wealth of $100,000
  - Faces a 25% chance of losing his automobile worth $20,000
  - Wealth with no theft (W_g) = $100,000 and probability of no theft = 0.75
  - Wealth with a theft (W_b) = $80,000 and probability of a theft = 0.25
  - Assume logarithmic utility

$$E(U) = 0.75U(W_g)+0.25U(W_b)=0.75 \ln W_g + 0.25 \ln W_b$$

$$E(U) = 11.45714$$

- The budget constraint
  - Written in terms of the prices of the contingent commodities

$$p_g W_g^* + p_b W_b^* = p_g W_g + p_b W_b$$

  - Assuming that these prices equal the probabilities of these two states

$$0.75(100{,}000) + 0.25(80{,}000) = 95{,}000$$

  - The expected value of wealth = $95,000

- The individual will move to the certainty line and receive an expected utility of

$$E(U) = \ln 95{,}000 = 11.46163$$

  - To be able to do so, the individual must be able to transfer \$5,000 in extra wealth in good times into \$15,000 of extra wealth in bad times
    - A fair insurance contract will allow this
    - The wealth changes promised by insurance

$$(dW_b/dW_g) = 15{,}000/\text{-}5{,}000 = -3$$

EXAMPLE 7.6 Insurance in the State-Preference Model

- A policy with a deductible provision
  - Insurance policy costs \$4,900, but requires the person to incur the first \$1,000 of the loss

$$W_g = 100{,}000 - 4{,}900 = 95{,}100$$

$$W_b = 80{,}000 - 4{,}900 + 19{,}000 = 94{,}100$$

$$E(U) = 0.75\ ln\ 95{,}100 + 0.25\ ln\ 94{,}100$$

$$E(U) = 11.46004$$

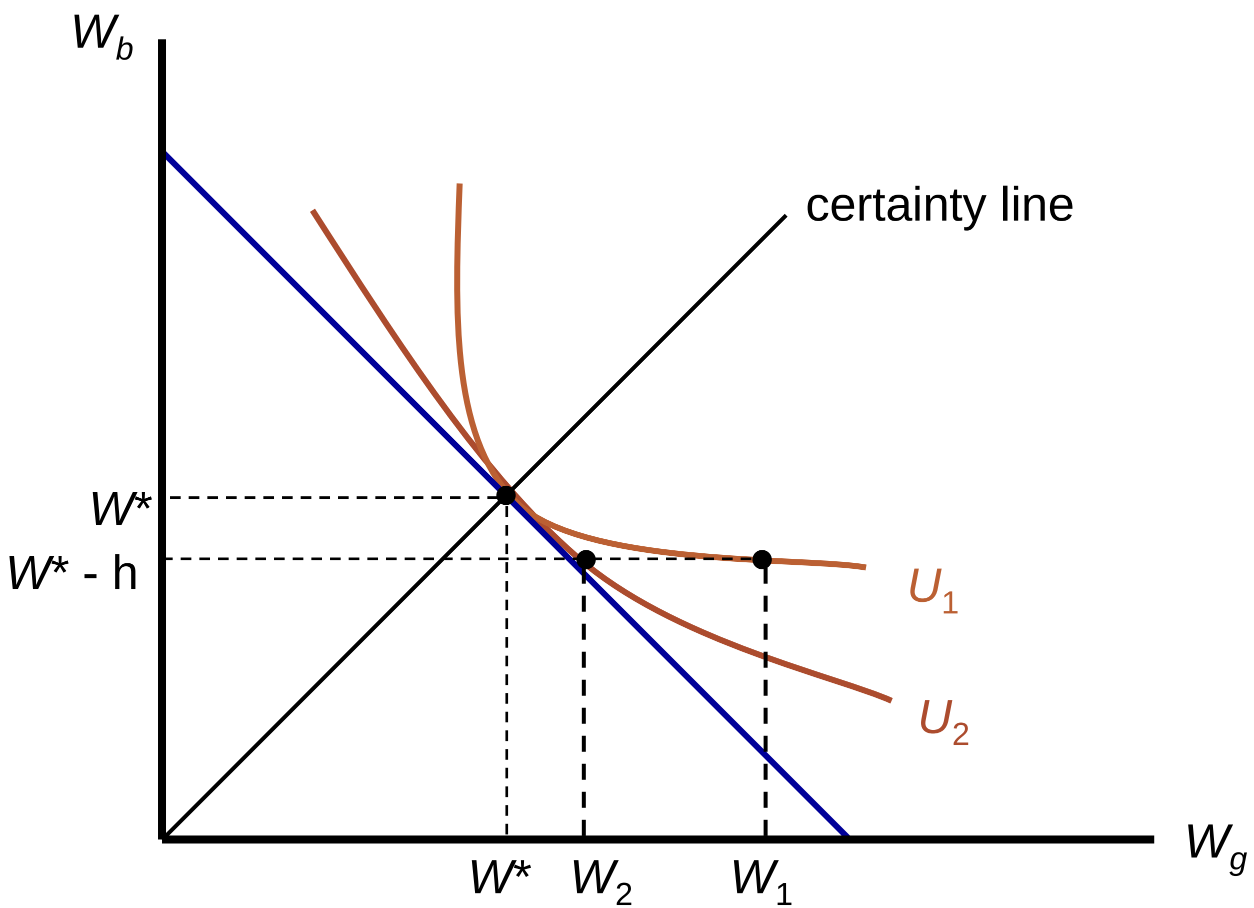
  - The policy still provides higher utility than doing nothing

Risk Premium

- Two people: same initial wealth of *W**
- Constant relative risk aversion: R

$$V(W_g, W_b) = \pi \frac{W_g^{\,R}}{R} + (1-\pi)\frac{W_b^{\,R}}{R}$$

Risk Aversion and Risk Premiums



Indifference curve U_1 represents the preferences of a risk-averse person, whereas the person with preferences represented by U_2 is willing to assume more risk. When faced with the risk of losing *h* in bad times, person 2 will require compensation of $W_2 – W^*$ in good times, whereas person 1 will require a larger amount given by W_1 - W^*.

Insurance

- Risk-aversion:
  - willing to pay a premium
  - always wants to buy full coverage a fair insurance
- Insurance market has problem because:
  - Large-scale disasters
  - Rare and unpredictable events
  - Informational disadvantage the company may have relative to the customer
    - Adverse selection problem
  - Moral hazard problem

Formal model of insurance

- Initial wealth: W
- Potential Loss: L with probably π
- Insurance premium per dollar coverage: p
- Consumer choose coverage: q

$$\max_q \pi U(W-L-pq+q)+(1-\pi)U(W-pq)$$

- FOC: $\pi U'(W-L-pq+q)(1-p)-p(1-\pi)U'(W-pq)=0$

$$\frac{U'(W-L+(1-p)q)}{U'(W-pq)}=\frac{1-\pi}{\pi}\frac{p}{1-p}$$

Formal model of insurance

- (Actuarial) fair insurance:

$$(1-\pi)pq - \pi(1-p)q = 0$$

- Then $p=\pi$
- Hence, $U'(W-L+(1-p)q) = U'(W-pq)$
- Under strict risk aversion ($U''<0$),

$$W-L+(1-p)q = W-pq$$

- Therefore, $q=L$.
- Full coverage!

The Portfolio Problem

- Basic model with one risky asset
  - Assume an individual has wealth (W_0) to invest in one of two assets
  - One asset yields a certain return of r_f
  - One asset's return is a random variable, r
  - k - the amount invested in the risky asset

EXTENSIONS

The Portfolio Problem

- The person's wealth at the end of one period

$$W = (W_0 - k)(1 + r_f) + k(1 + r)$$

$$W = W_0(1 + r_f) + k(r - r_f)$$

  - W is now a random variable: it depends on r
  - k can be positive or negative: can buy or sell short
  - k can be greater than W_0: the investor could borrow at the risk-free rate

The Portfolio Problem

- $U(W)$ - the investor's utility function
  - The von Neumann-Morgenstern theorem: he will choose k to maximize $E[U(W)]$
- The first-order condition:

$$\frac{\partial E\left[U\left(W\right)\right]}{\partial k}=\frac{\partial E\left[U\left(W_0\left(1+r_f\right)+k\left(r-r_f\right)\right)\right]}{\partial k}=$$

$$=E\left[U'\cdot\left(r-r_f\right)\right]=0$$

The Portfolio Problem

- As long as $E(r - r_f) > 0$
  - An investor will choose positive amounts of the risky asset
- As risk aversion increases
  - The amount of the risky asset held will fall
  - The shape of the *U'* function will change

CARA utility

- The investor's utility function - the CARA form: $U(W) = -\exp(-AW)$
  - Marginal utility function: $U'(W) = A\exp(-AW)$
  - End-of-period wealth:

$$U'(W) = A\exp[-A(W_0(1+r_f) + k(r - r_f))] =$$

$$= A\exp[-A(W_0(1+r_f)]\exp[-Ak(r - r_f)]$$

  - Optimality condition:

$$E[U'\cdot(r - r_f)] = A\exp[-AW_0(1+r_f)]$$

$$E[\exp(-Ak(r - r_f))\cdot(r - r_f)]=0$$

CARA utility

- CARA function
  - Implies that the fraction of wealth that an investor holds in risky assets should decrease as wealth increases
- CRRA form
  - All individuals with the same risk tolerance
    - Will hold the same fraction of wealth in risky assets
    - Regardless of their absolute levels of wealth

Portfolios of many risky assets

- Return on each of *n* risky assets
  - The random variable r_i (i = 1,…, n)
  - Expected values: $E(r_i)=\mu_i$
  - Variances: $Var(r_i) = \sigma^2_i$
  - An investor who invests a portion of his or her wealth in a portfolio of these assets will obtain a random return:

$$r_p = \sum_{i=1}^{n} \alpha_i r_i \quad \text{where} \quad \alpha_i \geq 0, \quad \sum_{i=1}^{n} \alpha_i = 1$$

EXTENSIONS

Portfolios of many risky assets

- Expected return on this portfolio

$$E(r_p) = \mu_p = \sum_{i=1}^{n} \alpha_i \mu_i$$

- If the returns of each asset are independent
  - The variance of the portfolio's return:

$$Var(r_p) = \sigma_p^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2$$

Optimal portfolios

- Solving the optimal portfolio problem
  - The first step: consider portfolios of just the risky assets
  - The second step: add in the riskless one
- Optimal portfolio of just the risky assets
  - Choose a general set of asset weightings (the α_i)
    - Minimize the variance (or standard deviation)
  - "Efficiency frontier" for risky asset portfolios

Optimal portfolios

- Add a risk-free asset
  - With expected return μ_f
  - And standard deviation $\sigma_f = 0$
- Optimal portfolios
  - "Market portfolio" consisting of all capital assets held in proportion to their market valuations
    - Expected return μ_m
    - Standard deviation σ_m

Optimal portfolios

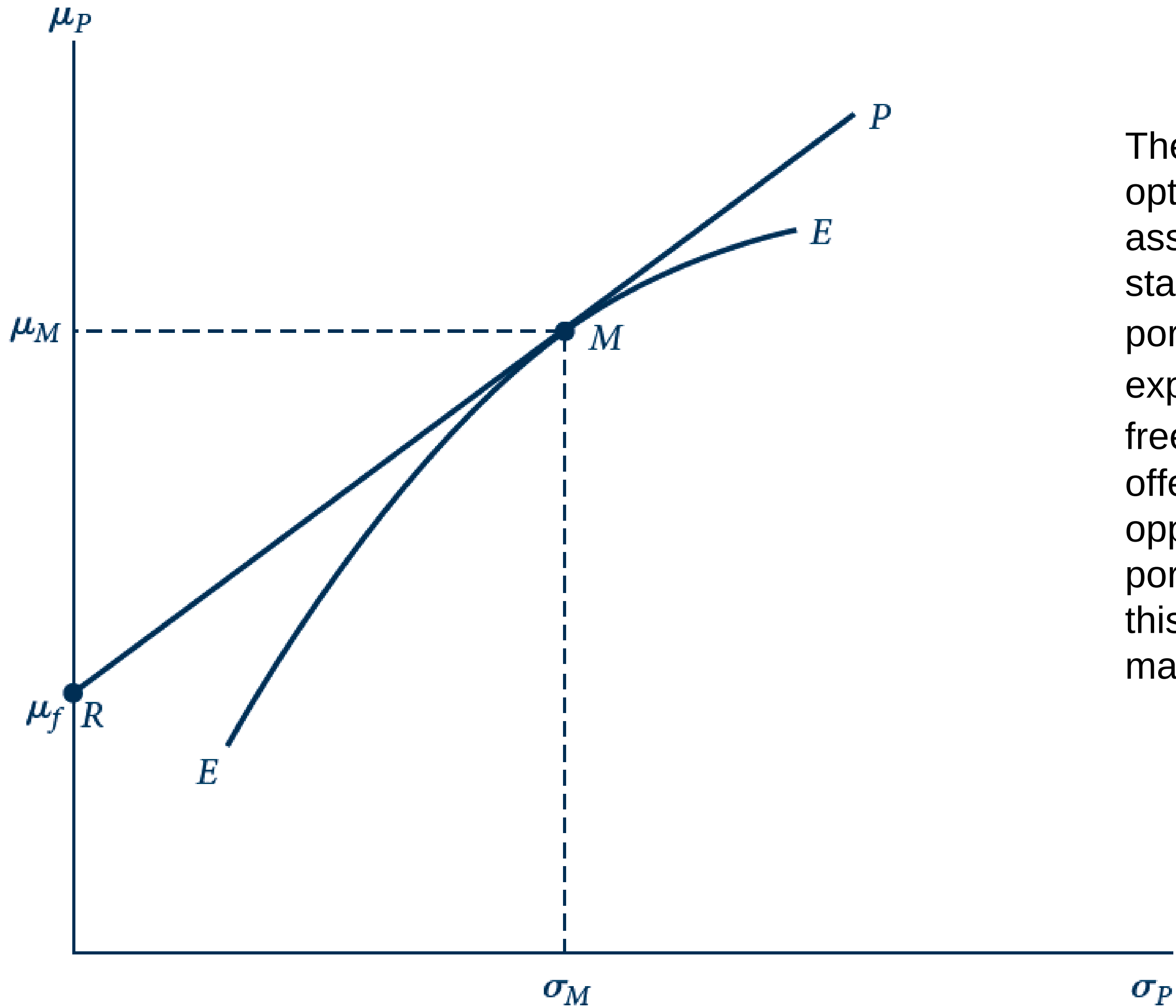
- Mixed portfolio (line RP)

$$\mu_p = \mu_f + \frac{\mu_m - \mu_f}{\sigma_M} \cdot \sigma_p$$

  - Permits individual investors to “purchase” returns in excess of the risk-free return (μ_M-μ_f) by taking on proportionally more risk (σ_p/σ_M)
  - Points to the left of the market point M: $\sigma_p/\sigma_M<1$ and $\mu_f < \mu_p < \mu_M$
  - High-risk points to the right of M: $\sigma_p/\sigma_M>1$ and $\mu_p > \mu_M$

FIGURE E 7.1

Efficient Portfolios



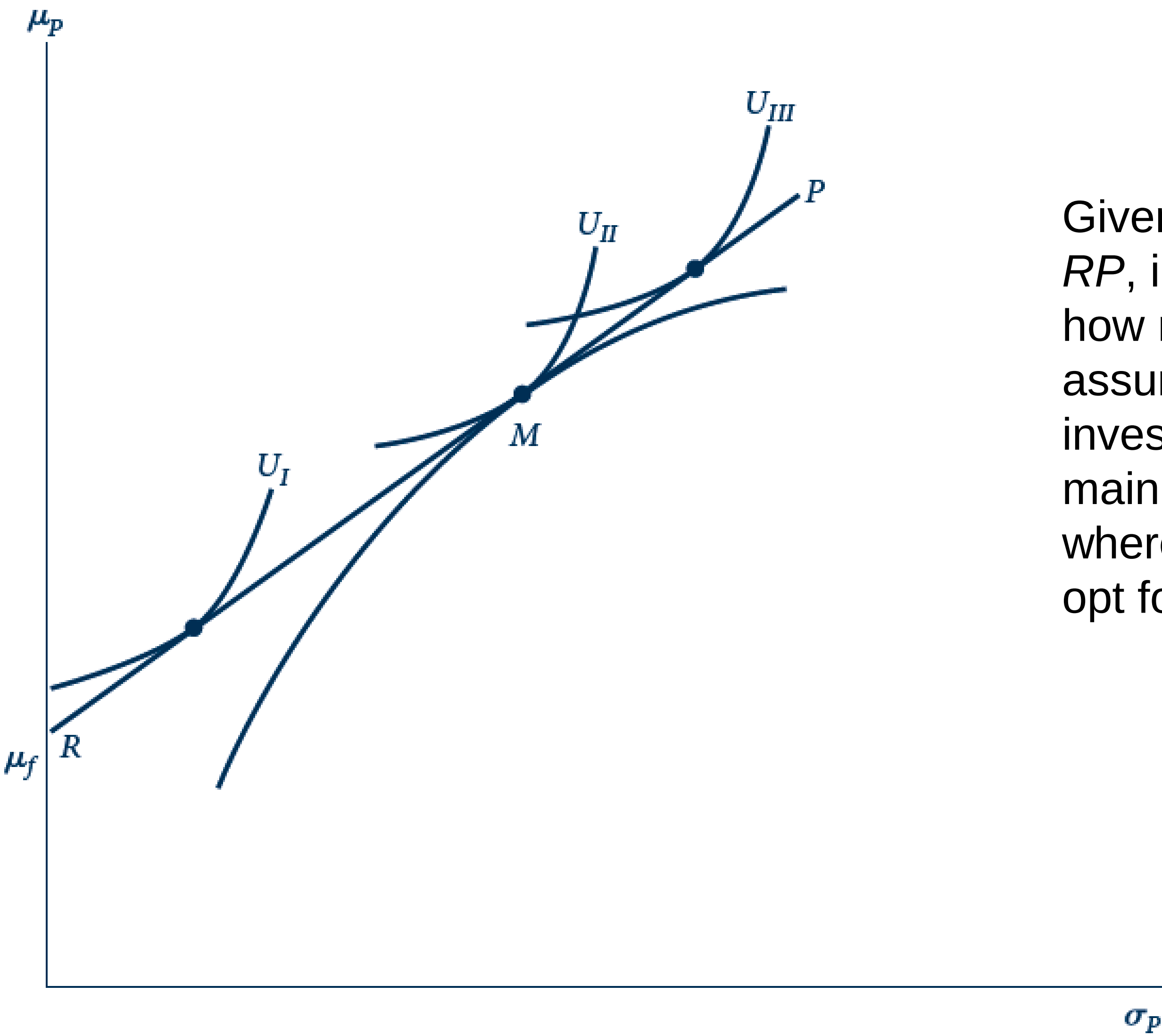
The frontier *EE* represents optimal mixtures of risky assets that minimize the standard deviation of the portfolio, σ_P, for each expected return, μ_P. A risk-free asset with return μ_f offers investors the opportunity to hold mixed portfolios along *RP* that mix this risk-free asset with the market portfolio, *M*.

Individual choices

- Individuals with low tolerance for risk (I)
  - Opt for portfolios that are heavily weighted toward the risk-free asset
- Investors willing to assume a modest degree of risk (II)
  - Opt for portfolios close to the market portfolio
- High-risk investors (III)
  - Opt for leveraged portfolios

FIGURE E 7.2

Investor Behavior and Risk Aversion



Given the market options *RP*, investors can choose how much risk they wish to assume. Very risk-averse investors (U_I) will hold mainly risk-free assets, whereas risk takers (U_{III}) will opt for leveraged portfolios.

Mutual funds

- Mutual funds
  - Pool the funds of many individuals
  - Able to achieve economies of scale in transactions and management costs
    - Fund owners to share in the fortunes of a much wider variety of equities
  - Managers have incentives of their own
  - Portfolios they hold may not always be perfect representations of the risk attitudes of their clients

Capital asset pricing model

- Portfolio
  - Small amount (α) of an asset with a random return x
  - Market portfolio, random return M
  - Return on the portfolio: $z = \alpha x + (1 - \alpha) M$
  - Expected return:

$$\mu_z = \alpha \mu_x + (1 - \alpha) \mu_M$$

  - Variance:

$$\sigma^2_z = \alpha^2 \sigma^2_x + (1 - \alpha)^2 \sigma^2_M + 2\alpha(1 - \alpha)\, \sigma_{x,M}$$

EXTENSIONS

Capital asset pricing model

$$\mu_z = \mu_f + (\mu_M - \mu_f) \cdot \frac{\sigma_z}{\sigma_M}$$

$$\frac{\partial \mu_z}{\partial \alpha} = \mu_x - \mu_M = \frac{\mu_M - \mu_f}{\sigma_M} \cdot \frac{\partial \sigma_z}{\partial \alpha}$$

$$\mu_x = \mu_f + (\mu_M - \mu_f) \cdot \frac{\sigma_{x,M}}{\sigma_M^2}$$