Notes on Offer Curve/Core

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1 Offer Curve and Core: typical question-solution

The assumptions of typical offer curve/contract curve problems are as follows.

- Pure exchange economy (so, no production).
- Edgeworth Box: there are two consumers who have continuous utility functions, u_1 and u_2 respectively.
- Initial endowments, (e_1, e_2), are given.

In what follows, let me make the following notations.

- p denotes the price ratio of good 1 and good 2, *i.e.* $p = \frac{p_1}{p_2}$.
- x_{li} denotes the amount of good l that consumer i consumes.
- e_{li} denotes the initial endowment of good l for consumer i.

1. **Step 1**: Consider p as given. Solve the utility maximization problem of each individual.

$$\begin{aligned} \max\ u_1(x_{11}, x_{21}) \quad \text{such that } px_{11} + x_{21} &\leq pe_{11} + e_{21} && (1) \\ \max\ u_2(x_{12}, x_{22}) \quad \text{such that } px_{12} + x_{22} &\leq pe_{12} + e_{22} && (2) \end{aligned}$$

   By solving above problems, we can get the Marshallian demand of each consumer, $x_{1i}(p, e_i)$ and $x_{2i}\,(p, ei)$.

2. **Step 2**: Construct an offer curve of each consumer. It is just the set of Marshallian demand of good 1 and good 2.

$$OC_i(x_{1i}(p, e_i), x_{2i}\,(p, e_i)) \tag{3}$$

   A competitive equilibrium is the point where OC_1 and OC_2 cross. The equilibrium price can be found in step 3.

3. **Step 3**: Find the equilibrium price p^*. Use the resource constraints at a competitive equilibrium to find p^*.

$$\begin{aligned} x_{11}(p^*) + x_{12}(p^*) &= e_{11} + e_{12} && (4) \\ x_{21}(p^*) + x_{22}(p^*) &= e_{21} + e_{22} && (5) \end{aligned}$$

4. **Step 4**: Find the set of Pareto Optima. At the Pareto optimal, MRS (ratio of marginal utility of each good) should be identical across individuals.

$$\frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial x_2}} = \frac{\frac{\partial u_2}{\partial x_1}}{\frac{\partial u_2}{\partial x_2}} \tag{6}$$

   An allocation that satisfies above condition is Pareto Optimal. So, find a set of allocations that satisfy equation (6).

5. **Step 5**: Find the core. Find a set of allocations that are in Pareto Optimal set and that any coalition of consumers does not block. In two-consumer case like Edgeworth, if a Pareto optimal allocation gives each consumer higher utility than the utility attained from initial endowment, the allocation is in the core.

2 Example 1

This exercise is the case of 3 consumers, but we can follow the similar procedure above.

Exercise 1 (2004 Winter Prelim Q1) *Consider a pure exchange economy with three consumers (labeled 1, 2 and 3) and three goods (labeled x, y, and z). Agent i's consumption vector is (x_i, y_i, z_i). Each agent is endowed with only one type of good: $\omega_1 = (1,0,0)$, $\omega_2 = (0,0,3)$ and $\omega_1 = (0,2,0)$ respectively. The consumers' preferences can be represented by utility functions as follows:*

$$\begin{aligned} U_1(x_1, y_1, z_1) &= x_1(y_1 + z_1) \\ U_2(x_2, y_2, z_2) &= 2x_2y_2 \\ U_3(x_3, y_3, z_3) &= 3x_3z_3 \end{aligned}$$

(a). Is the following allocation Pareto Optimal?

$$\begin{aligned} (x_1, y_1, z_1) &= (5/8, 1, 9/4) \\ (x_2, y_2, z_2) &= (1/4, 2, 0) \\ (x_3, y_3, z_3) &= (1/8, 0, 3/4) \end{aligned}$$

(b). Find a competitive equilibrium. Is it unique?
(c). Find the core of this economy.

2.1 (a) Pareto Optimality

Question (a) corresponds to step 4: Find the set of Pareto Optima. First, observe that $z_2 = 0$ and $y_3 = 0$ if an allocation is Pareto optimal. Otherwise, we can transfer $z_2 > 0$ or $y_3 > 0$ to the consumer 1 and he will be better-off holding other consumers' utility constant. Thus, the conditions of Pareto optimality can be reduced to:

$$\frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial y_1}} = \frac{\frac{\partial u_2}{\partial x_2}}{\frac{\partial u_2}{\partial y_2}} \to \frac{y_1 + z_1}{x_1} = \frac{y_2}{x_2}$$

and

$$\frac{\frac{\partial u_1}{\partial x_1}}{\frac{\partial u_1}{\partial z_1}} = \frac{\frac{\partial u_3}{\partial x_3}}{\frac{\partial u_3}{\partial z_3}} \to \frac{y_1 + z_1}{x_1} = \frac{z_3}{x_3}$$

Thus, at the Pareto optimal, it should be

$$\frac{y_1 + z_1}{x_1} = \frac{y_2}{x_2} = \frac{z_3}{x_3}$$

Clearly, $\frac{y_2}{x_2} = 8 \neq \frac{z_3}{x_3} = 6$. Thus, this allocation is not Pareto optimal.

2.2 (b) Find a competitive equilibrium

Question (b) corresponds to step 1 to 3.

1. **Step 1**: Solve each consumer's utility maximization problem assuming prices are given. Let's normalize the price of good z ($p_z = 1$).

$$\max\ x_1(y_1 + z_1) \qquad \text{such that } p_x x_1 + p_y y_1 + z_1 \le p_x$$

$$\max\ 2x_2y_2 \qquad \text{such that } p_x x_2 + p_y y_2 \le 3$$

$$\max\ 3x_3z_3 \qquad \text{such that } p_x x_3 + z_3 \le 2p_y$$

Solving these problems, we get the Marshallian demand as,

$$x_1 = \frac{1}{2}, y_1 = 0, z_1 = \frac{p_x}{2} \qquad if \quad p_y > 1$$

$$x_1 = \frac{1}{2}, y_1 = y_1, z_1 = \frac{p_x}{2} - y_1 \qquad if \quad p_y = 1$$

$$x_1 = \frac{1}{1+p_y}, y_1 = \frac{p_x}{1+p_y}, z_1 = 0 \qquad if \quad p_y < 1$$

$$x_2 = \frac{3}{2p_x}, y_2 = \frac{3}{2p_y}$$

$$x_3 = \frac{p_y}{p_x}, z_3 = p_y$$

2. **Step 2**: Construct an offer curve of each consumer. An offer curve is a set of Mashallian demands of a consumer.

$$OC_1 : \left(\frac{1}{2}, 0, \frac{p_x}{2}\right) \quad \text{if } p_y > 1$$

$$\left(\frac{1}{2}, y_1, \frac{p_x}{2} - y_1\right) \quad \text{if } p_y = 1$$

$$\left(\frac{1}{1+p_y}, \frac{p_x}{1+p_y}, 0\right) \quad \text{if } p_y < 1$$

$$OC_2 : \left(\frac{3}{2p_x}, \frac{3}{2p_y}, 0\right)$$

$$OC_3 : \left(\frac{p_y}{p_x}, 0, p_x\right)$$

3. **Step 3**: Find a competitive price p^*. Use the resource constraints to find the price. First, I rule out $p_y > 1$ and $p_y < 1$ by showing that some resource constraints are not satisfied.

   (a) $p_y > 1$. Then, $y_1 = y_3 = 0$, and $y_2 = \frac{3}{2p_y}$. Thus, according to resource constraints, it must be,

   $$y_2 = \frac{3}{2p_y} = 2 = \overline{y} \rightarrow p_y = \frac{3}{4}$$

   It contradicts that $p_y > 1$.

   (b) $p_y < 1$. Then $z_1 = z_2 = 0$, and $z_3 = p_y$. Thus, according to resource constraints, it must be,

   $$z_3 = p_y = 3 = \overline{z}$$

   It contradicts that $p_y < 1$.

(c) $p_y = 1$. Since $p_y \neq 1$ does not satisfy the resource constraints, it must be $p_y = 1$ at an equilibrium. We can pin down the price and allocation according to resource constraints.

$$x : \frac{1}{2} + \frac{3}{2p_x} + \frac{p_y}{p_x} = 1 \rightarrow p_x = 5$$

$$y : y_1 + \frac{3}{2p_y} = 3 \rightarrow y_1 = \frac{3}{2}$$

$$z : \frac{p_x}{2} - y_1 + p_x = 2$$

Thus, the competitive price and allocation is,

$$p^* = (5, 1, 1)$$

$$(x_1, y_1, z_1) = \left(\frac{1}{2}, \frac{3}{2}, 1\right)$$

$$(x_2, y_2, z_2) = \left(\frac{3}{10}, \frac{3}{2}, 0\right)$$

$$(x_3, y_3, z_3) = \left(\frac{1}{5}, 0, 1\right)$$

And the allocation is unique.

2.3 (c) Find the core.

By definition, any allocation that is in the core is Pareto Optimal. I will show that, in this question, any Pareto optimal allocation is in the core.

1. Coalition of one consumer. First, observe that $U_1(\omega_1) = U_2(\omega_2) = U_3(\omega_3) = 0$ at the initial endowment. Thus, any coalition of single consumer does not block any allocation.
2. Coalition of two consumers. Consider the coalition of consumer 1 and 2. In this coalition, neither consumer has good y. Thus, $U_2 = 0$ in any allocation of this coalition. Thus, the coalition of consumer 1 and 2 does not block any allocation. Similarly, any coalition of any two consumers does not block any allocation.
3. Coalition of three consumers. In this case, if the allocation is not Pareto optimal, then this coalition blocks that allocation. Otherwise, the coalition does not block.

According to 1-3, any Pareto optimal allocation is in the core.The set of Pareto optimal allocation is that

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ y_1 + y_2 &= 2 \\ z_1 + z_3 &= 3 \end{aligned}$$

$$\begin{aligned} \frac{y_1 + z_1}{x_1} &= \frac{y_2}{x_2} \quad \text{if } x_1 > 0 \text{ and } x_2 > 0 \\ \frac{y_1 + z_1}{x_1} &= \frac{z_3}{x_3} \quad \text{if } x_1 > 0 \text{ and } x_3 > 0 \\ \frac{y_2}{x_2} &= \frac{z_3}{x_3} \quad \text{if } x_2 > 0 \text{ and } x_3 > 0 \end{aligned}$$

3 Example 2 (non-increasing utility)

Exercise 2 (2002 Winter Prelim Q6) *Consider a two-person, two-good pure exchange economy with free disposal, in which each consumer has circular indifference curves. The utility function of consumer i is*

$$u^i(x,y) = -(x-\theta_i)^2 - (y-\theta_i)^2$$

where x is the amount of the first good consumed, y is the amount of the second good consumed, $i \in \{1,2\}, \theta_1 = 1$, and $\theta_2 = \frac{1}{4}$. Consumer 1 is endowed with 1 unit of x and zero units of y, and consumer 2 is endowed with 1 unit of y and zero units of x.

(a). Find the set of Pareto optimal allocations.
(b). Find the offer curve for consumer 1.
(c). Find a competitive equilibrium, and determine whether it is Pareto optimal.
(d). Is the competitive equilibrium unique?
(e). Does the second welfare theorem hold for this economy?

3.1 (a) Pareto optimality

Observe that consumer 2's preference is locally satiated at $\left(\frac{1}{4},\frac{1}{4}\right)$. Thus, at any Pareto optimal allocation, it must be $x_2 \le \frac{1}{4}$ and $y_2 \le \frac{1}{4}$. Otherwise, consumer 2 can transfer $x_2 - \frac{1}{4}$ or $y_2 - \frac{1}{4}$ to consumer 1 and both will be better off. Except that, step 4 works well to find Pareto optimal allocations. At Pareto optima, MRS (the ratio of marginal utility) should be identical across consumers.

$$\begin{aligned}\frac{\partial u^i}{\partial x_i} &= -2(x_i-\theta_i)\\ \frac{\partial u^i}{\partial y_i} &= -2(y_i-\theta_i)\end{aligned}$$

Thus,

$$\frac{\frac{\partial u^1}{\partial x_1}}{\frac{\partial u^1}{\partial y_1}} = \frac{x_1-1}{y_1-1} = \frac{x_2-\theta_2}{y_2-\theta_2} = \frac{\frac{\partial u^2}{\partial x_2}}{\frac{\partial u2}{\partial y_2}} \tag{7}$$

Plug resource constraints ($x_1+x_2=1$ and $y_1+y_2=1$) into equation (7). It is reduced to,

$$\frac{x_2}{y_2} = \frac{x_2-\theta_2}{y_2-\theta_2}$$

Thus, we get $x_2 = y_2$. The set of Pareto optimal allocations is

$$\begin{aligned}x_1+x_2 &= 1\\ y_1+y_2 &= 1\\ x_2 &= y_2 \le \frac{1}{4}\end{aligned}$$

3.2 (b) Offer curve

Follow step 1 and step 2. First, solve consumer i's problem:

$$\max\ -(x_i-\theta_i)^2 - (y_i-\theta_i)^2$$

$$s.t.\ px_i + y_i \le m_i$$

$$FOC \mid x_i: \quad -2(x_i-\theta_i) - \lambda p = 0$$

$$FOC \mid y_i: \quad -2(y_i-\theta_i) - \lambda = 0 \to \lambda = -2(y_i-\theta_i)$$

Solving these FOCs with regards to x and y, we get the Marshallian demands,

$$x_i = \frac{pm_i + \theta_i(1-p)}{1+p^2}$$

$$y_i = \frac{m_i + p\theta_i (p-1)}{1+p^2}$$

Thus, we can get consumer 1's offer curve by plugging $m_1 = p$ and $\theta_1 = 1$ into the Marshallian demands.

$$OC_1 = \left\{ \begin{array}{cc} \frac{p^2-p+1}{1+p^2}, & \frac{p^2}{1+p^2} \end{array} \right\}$$

3.3 (c) Competitive equilibrium

Again, notice that consumer 2's preference is locally satiated at $\left(\frac{1}{4}, \frac{1}{4}\right)$. With free disposal, consumer 2 is indifferent (and her utility is maximized) when $x_2 \geq \frac{1}{4}$ and $y_2 \geq \frac{1}{4}$. Thus, consumer 2 has "thick" indifference curve in the region $\left\{ \begin{array}{cc} \frac{1}{4} \leq x_2 \leq 1, & \frac{1}{4} \leq y_2 \leq 1 \end{array} \right\}$. (See MWG p.550 Figure 16.C.1.) In general, there are several equilibria with "thick" preference. I will follow a slightly modified step 3 to find one. Since consumer 2 is indifferent if $x_2 \geq \frac{1}{4}$ and $y_2 \geq \frac{1}{4}$, we can write the resource constraints as,

$$\begin{aligned} \frac{p^2-p+1}{1+p^2} + \frac{1}{4} &\leq 1 \\ \frac{p^2}{1+p^2} + \frac{1}{4} &\leq 1 \end{aligned}$$

Solving these inequalities, we get $2 - \sqrt{3} \leq p^* \leq \sqrt{3}$. To fine an allocation, choose any $p^* \in \left[2-\sqrt{3}, \sqrt{3}\right]$. Set $p^* = 1$, we get,

$$\begin{aligned} p^* &= 1 \\ (x_1, y_1) &= \left(\frac{1}{2}, \frac{1}{2}\right) \\ (x_2, y_2) &= \left(\frac{1}{2}, \frac{1}{2}\right) \end{aligned}$$

According to free disposal, consumer 2's utility is same as $u_2\left(\frac{1}{4}, \frac{1}{4}\right)$. However, this is not Pareto optimal. If consumer 2 transfer $\frac{1}{4}$ of each good to consumer 1, then consumer 1 will be better off holding consumer 2's utility.

3.4 (d) Uniqueness

As I discussed, the equilibrium is not unique. If we choose $p^* = \sqrt{3}$, then the following allocation is a competitive equilibrium.

$$\begin{aligned} p^* &= \sqrt{3} \\ (x_1, y_1) &= \left(\frac{4-\sqrt{3}}{4}, \frac{3}{4}\right) \\ (x_2, y_2) &= \left(\frac{\sqrt{3}}{4}, \frac{1}{4}\right) \end{aligned}$$

3.5 (e) 2nd welfare theorem

Since consumer 2's utility is not increasing (it is locally satiated), 2nd welfare theorem does not necessarily hold.