EC4101
Topic 4: Uncertainty

KO Chiu Yu

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1 Introduction

In this topic, we cover how an individual makes decision under uncertainty

1.1 Reading:

1. Snyder and Nicholoson, Chapter 7, Microeconomic Theory: Basic Principles and Extensions, 11th edition, 2012
2. Pratt, J. W., "Risk aversion in the small and in the large," Econometrica 32, January–April 1964, 122–136.

2 Uncertainty

Lottery $h = (x, y; p, 1 - p)$: wealth is x with probability p and y with probability $1 - p$.
Expected value of lottery $Eh = px + (1 - p)\, y$
Fair lottery: $E(h) = 0$
Is people always willing to participate fair lottery?
St. Petersburg paradox
A coin is flipped until a head appears If a head appears on the nth flip, the player is paid $\$2^n$

$$Eh = \sum_{i=1}^{\infty} 2^i \left(\frac{1}{2}\right)^i = \infty$$

Hence, utility of lottery is NOT the expected value of lottery.

2.1 Expected Utility

von Neumann-Morgenstern Theorem: If consumer's preference satisfies the below four axioms, the consumer preference can be represented by a utility function
Completeness: for any lottery x, y; either xRy, yRx or both
Transitivity: If xRy and yRz, then xRz
Continuity: If $xRyRz$, then there exists $0 \leq p \leq 1$ such that $px+(1-p)z$ is indifferent with y
Independence: If xRy, then for any lottery z, and $0 \leq p \leq 1$, we have $px + (1-p)zRpy + (1-p)z$
Cardinal utility: difference matters.
Under the theorem, individuals act as if they are maximizing expected utility.
St. Petersburg game may converge to a finite expected utility value if utility is concave.

2.2 Risk Attitude

Risk attitude

1. Risk-loving: willing to accept fair bet and even unfair bet

2. Risk-neutral: indifferent to fair bet

3. Risk-aversed: refuse fair bet

St. Petersburg paradox: most of us are risk-aversed
Natural measure of risk attitude: risk premium:
(Absolute) Risk Premium ($RP(h, W)$): amount needed to take lottery h given wealth W

$$U(W + E(h) - RP) = EU(W + h)$$

(Absolute) Certainty equivalent ($CE(h, W)$) of lottery $h = (x, y; p, 1-p)$ is

$$U(CE) = EU(W + h) = pU(W + x) + (1 - p)U(W + y)$$

By construction, $U(CE) = U(W + E(h) - RP)$, so

$$CE = W + E(h) - RP$$

Relative Risk Premium ($RRP(h, W)$): relative amount of wealth needed to take lottery hW given wealth W

$$U(E(hW) - W \times RRP) = EU(hW)$$

By rewriting, we have

$$W \times RRP(h, W) = RP(hW, W)$$

Risk Premia depends on lottery. We can decompose risk premium into riskiness of lottery (variance of lottery) and risk preference (utility):
Absolution risk premium: Arrow-Pratt's absolute risk aversion (ARA)

$$r(W) = -\frac{U''(W)}{U'(W)}$$

Proof. Definition of risk

$$E[U(W + h)] = U(W + E(h) - ARP(h; W))$$

Consider fair bet $E(h) = 0$ and taylor series expansion:

$$LHS : U(W - ARP) = U(W) - ARP \times U'(W) + \ldots$$

$$\begin{aligned} RHS : E[U(W + h)] &= E[U(W) - hU'(W) + \frac{h^2}{2} U''(W) + \ldots] \\ &= U(W) - E(h)U'(W) + E\left(\frac{h^2}{2}\right) U''(W) + \ldots \\ &= U(W) + \frac{Var(h)}{2} U''(W) + \ldots \end{aligned}$$

Dropping higher order terms, we have

$$
\begin{aligned}
U(W) - ARP \times U'(W) &= U(W) + \frac{Var(h)}{2} U''(W) \\
ARP\,(h, W) &= -\frac{U'(W)}{U''(W)} \times \frac{Var(h)}{2} \\
&= \frac{Var(h)}{2} r\,(W)
\end{aligned}
$$

Relative size: Arrow-Pratt's relative risk aversion (RRA)

$$
rr\,(W) = -\frac{WU''\,(W)}{U'\,(W)}
$$

Proof. By definition, we have $ARP(Wh; W) = W \times RRP(h; W)$ so that we have

$$
\begin{aligned}
RRP(h; W) &= \frac{1}{W} ARP(Wh; W) \\
&= -\frac{1}{W} \frac{U'(W)}{U''(W)} \times \frac{Var(Wh)}{2} \\
&= -\frac{WU'(W)}{U''(W)} \times \frac{Var(Wh)}{2} \\
&= \frac{Var(h)}{2} rr\,(W)
\end{aligned}
$$

Mean-variance preference:
Quadratic Utility: $U(W) = aW - bW^2$.

$$
\begin{aligned}
& EU(W + h) \\
= \; & aE(W + h) - bE((W + h)^2) \\
= \; & a(W + E(h)) - bE(W^2 + 2Wh + h^2) \\
= \; & a(W + E(h)) - b[W^2 + 2WE\,(h) + Var\,(h) + (E\,(h))^2]
\end{aligned}
$$

so it only depends on mean and variance of lottery.
Constant absolute aversion and lottery follows normal distribution ($U(w) = -e^{-rW}$)

When h follows normal distribution with mean μ and variance σ^2, the pdf is

$$f(h) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(h-\mu)^2}{2\sigma^2}\right)$$

Expected utility will be

$$\begin{aligned}
& E(U(W+h)) \\
= & \int_{-\infty}^{\infty} U(W+h) f(h)\, dh \\
= & \int_{-\infty}^{\infty} -e^{-r(W+h)} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(h-\mu)^2}{2\sigma^2}} dh \\
= & -e^{-rW} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{2rh\sigma^2+(h-\mu)^2}{2\sigma^2}} dh \\
= & -e^{-rW} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{2\mu r\sigma^2+r^2\sigma^4+\left(h-\mu+r\sigma^2\right)^2}{2\sigma^2}} dh \\
= & -e^{-\left(rW+\mu r-\frac{1}{2}r^2\sigma^2\right)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(h-\mu+r\sigma^2\right)^2}{2\sigma^2}} dh \\
= & -e^{-\left(rW+\mu r-\frac{1}{2}r^2\sigma^2\right)} = U\left(W+\mu-\frac{1}{2}r\sigma^2\right)
\end{aligned}$$

2.3 Application

2.3.1 Diversification

Consider there are two identical stocks with same means μ and same variances σ^2 but zero covariance. Then invest all money in one of two stocks gives expected return of μ and variance σ^2. However, if the investor splits α porton of the money into one stock and $(1-\alpha)$ to another stock. The expected return is still μ but variance is $\left(\alpha^2+(1-\alpha)^2\right)\sigma^2$. At the minimium, variance of the portfolio is $\sigma^2/2$ when $\alpha = 1/2$.

2.3.2 Contingent Commodity

Two states of the world: good states and bad states. Each state is associated with one good: W_g and W_b. Probability of good state is π and that of bad state is $1-\pi$. Hence, expected utility is

$$EU = \pi U(W_g) + (1-\pi) U(W_b)$$

and the budget constraint is $p_g W_g + p_b W_b \leq W$. Assuming interior solutions, we have

$$MRS = \frac{\partial EU/\partial W_g}{\partial EU/\partial W_b} = \frac{\pi U'(W_g)}{(1-\pi) U'(W_b)} = \frac{p_g}{p_b}$$

Under an actuarially fair market, price ratio and odd ratio are equal

$$\frac{p_g}{p_b} = \frac{\pi}{1-\pi}$$

so that

$$\frac{U'(W_g)}{U'(W_b)} = 1$$

By strict concavity of U, we have $W_g = W_b$. Hence, individuals will try to make themselves on the certainty line.

2.3.3 Insurance

An application of contingent commodity. Individual has initial wealth of W facing a potential loss of L with probability π. Insurance company offers insurance with premium p for per dollar coverage. The individual chooses coverage level q:

$$\max_q \pi U(W - L - pq + q) + (1-\pi) U(W - pq)$$

The first order condition implies

$$(1-p) \pi U'(W - L - pq + q) - p(1-\pi) U'(W - pq) = 0$$

so that

$$\frac{\pi}{1-\pi} \frac{U'(W - L - pq + q)}{U'(W - pq)} = \frac{p}{1-p}$$

The second order condition is

$$(1-p)^2 \pi U''(W - L - pq + q) + p^2 (1-\pi) U''(W - pq)$$

which is negative if $U'' < 0$.
Under actuarial fair market (competitive insurance market), expected profit of the insurance is zero:

$$(1-\pi) pq - \pi q (1-p) = 0$$

so that $p = \pi$ and hence

$$\frac{U'(W - L - pq + q)}{U'(W - pq)} = 1$$

By strict concavity of U, we have $W - L - pq + q = W - pq$, or $q = L$. Hence, risk-averse individual will take ful coverage under actaurial fair market.

A Review of probability

A.1 Descriptive Statistics

1. Descriptive statistics gives a clearer picture of the data: numerical or visual representation.

2. Three main types of numerical representation: central tendency, dispersion and shape. Central tendency is measuring where most data are. Dispersion is about how spread data are. It is also telling how accurate of measure of central tendency is. Shape includes skewness and kurtosis(peakedness).

3. Central tendency: mean ($\bar{x} = \sum_{i=1}^{n} x_i/n$), median (the middle data) and mode (most frequent data). Introduction of sequence and Summation notation.

   (a) Example 1: 1, 2, 3, 3, 4, 5. mean= median = mode = 3

   (b) Example 2: 1, 2, 3, 4, 5. median = 3.

   (c) Example 3: 1, 2, 3, 4, 5, 6. median = 3.5.

   (d) Other averages: weighted average ($\bar{x}_w = \sum_{i=1}^{n} x_i w_i / \sum_{i=1}^{n} w_i$), moving average, geometric mean, harmonic mean

4. Dispersion: range($\max x_i - \min x_i$), interquartile range IQR ($IQR = Q_3 - Q_1$) (Q_1 = median of 1st half data and Q_3 = median of 2nd half of data, definition of percentile and quartile), variance($\sigma^2 = \sum (x_i - \bar{x})^2 / n$), standard deviation ($\sigma = \sqrt{\sigma^2}$), five-number summary, coefficient of variation ($CV = \sigma/\bar{x}$)

   (a) Motivating example: (data set 1) 7, 8, 8, 9 and (data set 2) 6, 8, 8, 10. Both data sets have same mean, median and mode but data set 1 are more concentrated near around the mean.

(b) Numerical Example: 1, 2, 2, 3, 3, 3, 4, 4, 5. Mean = median = mode = 3, Range = 4, IQR = 2, variance = 4/3 and standard deviation = $2/\sqrt{3}$.

(c) Remark: In the textbook, IQR is calculated differently. The formula is $Q_1 = 0.25 \times (n+1)$th data and $Q_3 = 0.75 \times (n+1)$th data. Example: Data set is $2, 3, 4, 5, 6, 7$. $Q_1 = 0.25 \times (6+1) = 7/4$th data $= 2 \times \frac{1}{4} + 3 \times \frac{3}{4} = \frac{11}{4}$. However, in our definition, $Q_1 = 3$.

5. Shape: skewness $\left(\sum (x_i - \bar{x})^3 / n\right)$ measures how asymmetric data is and kurtosis $\left(\sum (x_i - \bar{x})^4 / n\right)$ tells thickness of the tail.

6. Appropriate visual representation depends on purpose of presentation and data source. Graphs: pie chart, bar chart, histogram, scatter plot, radar chart, stem-and-leave graph, box-plot, frequency table.

A.2 Probability Theory

1. Definition of Probability: Frequentist (long run relative ratio) versus Bayesian (subjective belief).

2. Basic Terminology:

   (a) Random Experiment: something is going to occur, but we don't know what will happen (recall that we are talking frequentist probability, for something to occur in the long run, we can define random experiment meaningfully as something can repeat itself for many times.)

   (b) Outcome: one possibility of random experiment

   (c) Sample space: all possible outcomes

   (d) Event: a group of outcomes

   (e) Example 1: Rolling a die. 1 is an outcome and 2, 3, 4, 5, 6 are also outcomes. Hence, the sample space $S = \{1, 2, 3, 4, 5, 6\}$. Even outcomes $E = \{2, 4, 6\}$ is an event, Odd outcomes $O = \{1, 3, 5\}$ is an event, and Large outcomes $L = \{4, 5, 6\}$ is also an event.

   (f) Example 2: rolling two dies: $S = \{(1,1), (1,2), \ldots, (6,6)\}$ where the first and second entries are the number of the first and the second dies. Total number of outcome is 36. The event "The sum of two dies is less than

4" consists of outcomes $(1,1)$,$(1,2)$ and $(2,1)$. Clearly, $(1,2)$ and $(2,1)$ are different events.

(g) Permutation and Combination: they will simplify calculations of number of outcomes for complicated events but we will skip this part in this course.

3. Operation on Events:

   (a) Intersection: looking for common outcomes in events, like "AND", similar to finding "H.C.F." Example: $E \cap L = \{4,6\}$, $O \cap L = \{5\}$.

   (b) Union: looking for all outcomes in events, like "OR", similar to finding "L.C.M." Example: $E \cup L = \{2,4,5,6\}$, $O \cup L = \{1,3,4,5,6\}$.

   (c) Complement: looking for outcomes not in the event (but in sample space), like "NOT". Example: $E^C = \{1,3,5\} = O$, $\bar{L} = \{1,2,3\}$

4. Properties of events

   (a) Mutually Exclusive Events: cannot happen at the same time so that the interesection is empty. Example: $E \cap L = \phi$.

   (b) Collectively Exhaustive Events: union of all events IS the sample sapce. Example: $E \cup O = S$.

   (c) Mutually Exclusive and Exhaustive events: also call partition of sample space. We call $P_1, P_2, \ldots, P_n$ is a partition of sample sapce if they are mutually exclusive ($P_i \cap P_j = \phi$ for all $i \neq j$) and exhaustive ($P_1 \cup P_2 \cup \ldots \cup P_n = S$).

5. Axiomatic definitions of probability :

   (a) Axiom 1: for any event A, we have $0 \leq \Pr(A) \leq 1$.

   (b) Axiom 2: If event A happens if and only if one of outcomes $O_1, O_2, \ldots, O_n$ happens ($A = \{O_1, O_2, \ldots, O_n\}$), then we have

$$\Pr(A) = \sum_{i=1}^{n} \Pr(O_i).$$

   (c) Axiom 3: $\Pr(S) = 1$ where S is the sample space.

6. Example 1: rolling a fair die: $S = \{1, 2, 3, 4, 5, 6\}$. Since the die is fair, each outcome is equally likely. By axioms, it must be $\Pr(1) = \Pr(2) = \Pr(3) = \Pr(4) = \Pr(5) = \Pr(6) = 1/6$. By axiom 2, we have $\Pr(E) = \Pr(2) + \Pr(4) + \Pr(6) = 3/6 = 1/2$. (Remark: Of course, the die might be unfair so that certain outcome is more likely than others but it does not violate any axioms.)

7. Three axioms also implies three rules:

   (a) Complement rule: $\Pr(A) = 1 - \Pr\left(\bar{A}\right)$ or $\Pr(A) = 1 - \Pr(A)$ which is quite useful sometimes.

   (b) Addition Rule: $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cup B)$ can be jusified by drawing a diagram or from the Inclusion-exclusion principle in set theory.

   (c) Total probability rule: $\Pr(S) = \sum_{i=1}^{n} \Pr(P_i)$ if $P_1, P_2, \ldots, P_n$ is a partition of sample sapce. Clearly, it follows from addition rule. (We will use it a lot in Bayes theorem)

8. Odd ratio in favor of A: $Odds = \Pr(A) / \Pr\left(\bar{A}\right) = \Pr(A) / [1 - \Pr(A)]$. Special terms used in the gambling industry.

9. Conditional probability: how to update probability when we know something is true. Since the sample space changes, we have to update it

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

   which can be justified by a diagram or argument of counting number of outcomes. (Of course, we have assumed $\Pr(B) > 0$.)

10. Definition of Statisticaly Indenpendence: Events A and B are (statistically) independent if and only if

    (a) $\Pr(A \mid B) = \Pr(A)$ or $\Pr(B \mid A) = \Pr(B)$

    (b) $\Pr(A \cap B) = \Pr(A)\Pr(B)$ (given $\Pr(A) > 0$ and $\Pr(B) > 0$)

11. Multiplication Rule: just rearrange the definition of conditional probability

$$\begin{aligned}\Pr(A \cap B) &= \Pr(A \mid B)\Pr(B) \\ &= \Pr(B \mid A)\Pr(A)\end{aligned}$$

12. Bayes Theorem

$$\Pr(A \mid B) = \frac{\Pr(B \mid A)\Pr(A)}{\Pr(B)}$$

(a) Proof. From definition of condition probability,

$$\Pr(A \mid B) = \frac{\Pr(A \cap B)}{\Pr(B)}$$

and by multiplication rule, we have

$$\Pr(A \cap B) = (B \mid A)\Pr(A)$$

so that we have

$$\Pr(A \mid B) = \frac{(B \mid A)\Pr(A)}{\Pr(B)}$$

as requested.

(b) This is the central idea of learning. We update our information as we have new information. Clearly, even people have different beliefs at the beginning, as more new information coming out, ultimately, we will know the true state of the world. (Of course, more often the time, we don't have enough information coming out to judge, so we have disagreement.)

(c) Example: 10% of population has a certain disease. The only cure of disease is dangerous: can completely cure the diesase but kill the patient if he/she does not have the diease. A scientist invented a test with 90% accuracy: if the patienet has the disease, 90% of time the test gives positive results; and if the patienet is healthy, 90% of time the test gives negative results. Then a natural question: what is the probability that a person has the disease if the test result is positive?

You are given

$$\begin{aligned}
\Pr(\text{disease}) &= 0.1, \\
\Pr(\text{+ve} \mid \text{disease}) &= 0.9\text{, and} \\
\Pr(\text{-ve} \mid \text{no disease}) &= 0.1.
\end{aligned}$$

The question is asking for $\Pr(\text{disease} \mid +\text{ve})$. By Bayes theorem, we have

$$\Pr(\text{disease} \mid +\text{ve}) = \frac{\Pr(+\text{ve} \mid \text{disease}) \Pr(\text{disease})}{\Pr(+\text{ve})}$$

Note that we know everything in the numerator. For the denominator, we need to use the total probability rule, that is,

$$\Pr(+\text{ve}) = \Pr(+\text{ve} \mid \text{disease}) \Pr(\text{disease}) + \Pr(+\text{ve} \mid \text{no disease}) \Pr(\text{no disease})$$

For $\Pr(+\text{ve} \mid \text{no disease})$ and $\Pr(\text{no disease})$, we use complement rule that

$$\begin{aligned} \Pr(+\text{ve} \mid \text{no disease}) &= 1 - \Pr(\text{-ve} \mid \text{no disease}) = 0.9 \\ \Pr(\text{no disease}) &= 1 - \Pr(\text{disease}) = 0.9 \end{aligned}$$

Hence, we have

$$\Pr(\text{disease} \mid +\text{ve}) = \frac{(0.9)(0.1)}{(0.9)(0.1) + (0.9)(0.1)} = 0.5$$

so that even the test is highly accurate but positive result does not mean the person has high probability of getting the disease.

A.3 Random Variable

1. Random variable: Mapping outcomes to numbers. Why? Then we can describe probability of events in sample space easily because we have the tool "function" in mathematics which help us relate two different sets of numbers.

   (a) Example 1: Tossing a coin once. Sample sapce $S = \{\text{Head}, \text{Tail}\}$. Say we define X to be a random variable of total number of heads. Then outcome of head means $X = 1$ and outcome of tail means $X = 0$.

   (b) Example 2: Tossing a coin twice. Sample sapce $S = \{\text{HH, HT, TH, TT}\}$ where H means Head and T mean tail. Say we define X to be a random variable of total number of heads. Then $X = 0$ means TT, $X = 1$ means union of HT and TH and $X = 2$ means HH.

2. Discrete random variable: the number of outcome in the sample space is finite.

Then we can assign integers to each outcome in sample space.

(a) Probability distribution function:

$$p(x) = \Pr(X = x)$$

(b) Cumulative probability function:

$$F(x) = \Pr(X \leq x)$$

(c) Example 3. Rolling a fair die. Sample space $S = \{1, 2, 3, 4, 5, 6\}$. Let X be the random variable of face value of the die. Then X can be integers from 1 to 6 (inclusive).

X	1	2	3	4	5	6
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	$\frac{6}{6}$

Draw $p(x)$ and $F(x)$.

(d) Expectation value: weighted mean of random variable with probability distribution function as weights.

$$E(X) = \sum xp(x)$$

since $\sum p(x) = 1$. We often deonte $E(X) = \mu_X$.

(e) Variance:

$$Var(X) = \sum [x - E(X)]^2 p(x)$$

We often deonte $Var(X) = \sigma_X^2$.

(f) Example 3 (revisited). Rolling a fair die. Expected value of X is

$$\begin{aligned} E(X) &= 1\left(\frac{1}{6}\right) + 2\left(\frac{1}{6}\right) + 3\left(\frac{1}{6}\right) + 4\left(\frac{1}{6}\right) + 5\left(\frac{1}{6}\right) + 6\left(\frac{1}{6}\right) \\ &= 3.5 \end{aligned}$$

and variance of X is

$$\begin{aligned} Var(X) &= (1-3.5)^2\left(\frac{1}{6}\right)+(2-3.5)^2\left(\frac{1}{6}\right)+(3-3.5)^2\left(\frac{1}{6}\right) \\ &\quad +(4-3.5)^2\left(\frac{1}{6}\right)+(5-3.5)^2\left(\frac{1}{6}\right)+(6-3.5)^2\left(\frac{1}{6}\right) \\ &= 2.9167 \end{aligned}$$

(g) Properties of expectation and variance functions:

$$\begin{aligned} E(a+bX) &= a+bEX \\ Var(a+bX) &= b^2 VarX \end{aligned}$$

3. Continuous Random Variable: Mapping outcome to real numbers.

(a) Cumulative probability function:

$$F_Y(y)=\Pr(Y\leq y)=\int_{-\infty}^{y} f(v)\,dv$$

(b) Probability distribution function: (not needed)

$$f_Y(y)=\frac{dF_Y(y)}{dy}$$

(c) Expectation and Variance: (not needed)

$$\begin{aligned} E(Y) &= \int_{-\infty}^{\infty} yf(y)\,dy \\ Var(Y) &= \int_{-\infty}^{\infty}[y-E(Y)]^2 f(y)\,dy \end{aligned}$$

4. Commonly used discrete distributions: Bernoulli, Binomial, Possion, uniform (all skipped)

5. Commonly used continuous distributions: uniform, normal, Chi-square, student's t, F-distribution

(a) Uniform: each outcome in sample space is equally likely (skipped)

(b) Normal: Invented by Gauss as measurement of error. By Central Limit Theorem it is very common. Starndardization of normal random variable. Property of standard normal. Tables looking up.

Example 4: Let Z follow a standard normal distribution. Using the table, we can

(a) find $\Pr(Z < 1.4) = 0.9192$

(b) find $\Pr(-1.6 < Z < 1.4) = 0.9192 - (1 - 0.9452) = 0.8644$,

(c) find $\Pr(Z < -1.5) = 1 - 0.9332 = 0.0668$.

Conversevly, we can answer the following questions:

(d) The probability is 0.5 that Z is less than what number? 0

(e) The probability is 0.2 that Z is greater than what number? -0.845

(f) The probability is 0.5 that Z is in the symmetric interval about the mean between two numbers? -0.675, 0.675

Furthermoer, if X follows a normal distribution with $\mu = 20$ and $\sigma^2 = 16$. Then we can

(g) find the probability that X is greater than 24: $\Pr(X > 24) = \Pr(Z > 1) = 1 - 0.8413 = 0.1587$,

(h) find the probability that X is between than 16 and 24: $\Pr(16 < X < 24) = \Pr(-1 < Z < 1) = 0.8413 - (1 - 0.8413) = 0.6826$

(c) Chi-square: sum of squared of independent variables with standard normal distributions (skipped)

(d) Student's t, F-distribution: for hypothesis testings (skipped)