EC4101
Topic 5: Game Theory

KO Chiu Yu

November 20, 2012

1 Introduction

In this topic, we review non-cooperative game theory.

1.1 Reading:

1. Jean Tirole, Chapter 11, The Theory of Industrial Organization, 1988.
2. Bagwell, Kyle and Asher Wolinsky, "Game Theory and Industrial Organization", 2002. "Handbook of Game Theory with Economic Applications," volume 3, 2002.
3. Gibbons, R. A primer in game theory, Harvester-Wheatsheaf, 1992.
4. Gibbons, R. "An introduction to Applicable Game Theory," Journal of Economic Perspectives, 1997, 127-129.

1.2 Game Theory Problem and Decision Problem

Non-cooperative Game Theory:

1. interaction among economic agents – interactive situation
2. Situation is complicated so expected utility theorem is used to simplify the treatment

Example. Monopoly problem

1. Inverse demand curve: $P = P(Q)$
2. Cost curve: $C = C(Q)$
3. Firm's objective: maximize profit $\pi = P(Q) \times Q - C(Q)$
4. The decision of consumer is degenerated so it is a decision problem

Example. Duopoly problem

1. Two firms: firm 1 and firm 2
2. Firm 1: chooses production quantity Q_1 subject to cost function $c(Q_1)$, maximizes profit
$$\pi_1 = P(Q_1 + Q_2) \times Q_1 - C(Q_1)$$
3. Firm 2: chooses production quantity Q_2 subject to cost function $c(Q_2)$, maximizes profit
$$\pi_2 = P(Q_1 + Q_2) \times Q_2 - C(Q_2)$$
4. As each firms decision would affect another firm, it is a game
5. Note that when we have many firms so that $Q = Q_1 + \cdots + Q_n$ in the way that each firm is small compared to the whole market, then firm would just take price as given and then we have decision problem back

2 Static Game of Complete Information

2.1 Strategic-form game

A strategic-form game $G = \left(N, (S_i, u_i)_{i \in N}\right)$ is represented by three elements:

1. $N = \{1, \ldots, n\}$ is the set of players
2. $s_i \in S_i$ is the strategy set for player $i \in N$
3. u_i is the payoff function of player i which maps strategies profile $(s_1, \ldots, s_n)$ to a utility number
$$u_i : \prod_{j \in N} S_j \to \mathbb{R}$$

Taking the duopoly example, we have $N = \{1, 2\}$, $Q_i \in S_i = \mathbb{R}_+$ and $u_i(Q_1, Q_2) = P(Q_1 + Q_2) Q_i - C(Q_i)$.

2.2 Strict Dominance

Definition. For player $i \in \mathbb{N}$, $s_i \in S_i$, is **strictly dominated** by $s_i' \in S_i$ iff

$$u_i(s_i, s_{-1}) < u_i(s_i', s_{-i}), \quad \forall s_{-i} \in S_{-i}$$

Example. Prisoner dilemma

		2	
		C	D
1	C	-1,-1	-9,0
	D	0,-9	-6,-6

1. C is strictly dominated
2. Once C is removed from the strategy choice, the expected outcome is $\{D, D\}$

2.3 Iterated elimination of strictly dominated strategy (i.e.s.d.s.)

1. Since strict dominance implies higher payoff under every circumstance, rationality implies a strictly dominated strategy is not used.
2. Iterated strict dominance: eliminate strictly dominated strategies to get a smaller game, then repeat this procedure.
3. Requires strong rationality and common knowledge assumption.
4. Note that this is different from elimination of weakly dominated strategies. (Experimental results on "Guess 2/3 of the average") [weak dominance never a lower payoff no matter what the opponent does, and sometimes a higher payoff]

Example Solvable by i.e.s.d.s.

			2	
		L	M	R
1	U	1,0	1,2	0,1
	D	0,3	0,1	2,0

1. R is strictly dominated by M
2. Once R is eliminated, D is strictly dominated by U
3. Once D is eliminated, L is strictly dominated by M
4. Hence, $\{U, M\}$ should be the equilibrium outcome.

Example V: Not solvable by i.e.s.d.s.

		2		
		L	M	R
	T	0,4	4,0	5,3
1	M	4,0	0,4	5,3
	D	3,5	3,5	6,6

1. There is no strictly dominated strategy
2. So not solvable by i.e.s.d.s.

2.4 Nash Equilibrium

Definition. For a normal form game $G = \left(N, (S_i, u_i)_{i \in N}\right)$, a strategy profile $s^* \in \prod_{j \in N} S_j$ is a **Nash equilibrium** if and only if

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right), \quad \forall s_i \in S_i, \forall i \in N$$

A strategy profile is a Nash equilibrium if for every player, there is no incentive to unilaterally deviate from the proposed strategy given other players keeping the proposed strategy.

Definition. A strategy s_i is a **best response** with respect to all other players' strategies s_{-i} if

$$u_i\left(s_i, s_{-i}\right) \geq u_i\left(s_i', s_{-i}\right), \quad \forall s_{-i} \in S_{-i}$$

Definition. The set of best responses for player i given s_{-i} is the **best response Correspondence**

$$\beta_i\left(s_{-i}\right) = \{s_i \in S_i \mid u_i\left(s_i, s_{-i}\right) \geq u_i\left(s_i', s_{-i}\right), \quad \forall s_{-i} \in S_{-i}\}$$

Claim. Nash equilibrium is a fixed point:

Rewrite the definition of Nash equilibrium, for all $i \in N$, we have

$$s_i^* \in \beta_i(s_{-i}^*)$$

Hence, we have

$$s^* = (s_1^*, s_2^*, \ldots, s_n^*) \in \left(\beta_1(s_{-1}^*), \beta_2(s_{-2}^*), \ldots, \beta_n(s_{-n}^*)\right)$$

or compactly,

$$s^* \in \beta(s^*)$$

where $\beta_i : S_{-i} \twoheadrightarrow S_i$.

Example. Battle of Sexes

		girl	
		opera	football
boy	opera	5,3	2,2
	football	2,2	3,5

1. Note that the payoff is given by the rule that unfavorable event have 0 payoff, together have 2 payoff and favorite one has 3 payoff
2. Two NEs $= \{\{\text{opera, opera}\}, \{\text{football, football}\}\}$

Example Chicken Game

		2	
		straight	swerve
1	straight	-1,-1	10,0
	swerve	0,10	-3,-3

1. Two NEs $= \{\{\text{serve, straight}\}, \{\text{straight, swerve}\}\}$
2. Multiple NEs
3. Might not be best for either player; So NE can be inefficient

Hence, there is no speical property related to Nash equilibirum.

2.4.1 Iterated elimination and Nash equilibrium:

Proposition. Suppose G is a finite game, If $s^* \in \prod_{j \in N} S_j$ is the unique survivor of i.e.s.d.s., then s^* is a Nash equilibrium. (Proof in appendix)
Proposition. If s^* is a Nash equilibrium, then s^* survives i.e.s.d.s. (Proof in appendix)

2.5 Application: Cournot Game

Cournot Problem in linear and symmetric demand with two firms:

1. Inverse demand $P(q_1, q_2) = a - q_1 - q_2$ where $a > 0$
2. Maringal cost of firm is zero.
3. Profit of firm $i = 1, 2$: $\pi_i(q_i, q_j) = [a - (q_i + q_j)]\, q_i$

To write it in normal form:
Player set $N = \{1, 2\}$
Strategy: $q_1 \geq 0, q_2 \geq 0$.
Payoff functions $u_1 = \pi_1(q_1, q_2)$, $u_2 = \pi_2(q_1, q_2)$
To find NE, we have to find best response correspondence of firm 1. (firm 2 is similar)
Given q_2, recall the objective for firm 1 is

$$\max_{q_1} \pi_1(q_1, q_2) = [a - (q_1 + q_2)]\, q_1$$

The FOC is

$$\frac{\partial \pi_1}{\partial q_1} = 0 \Rightarrow a - 2q_1^* - q_2 = 0$$

Hence, the best response is

$$\beta_1(q_2) = q_1^* = \frac{a - q_2}{2}$$

By symmetry, we have

$$\beta_2(q_1) = q_2^* = \frac{a - q_1}{2}$$

Remember Nash equilibirum requires

$$q_1^* = \beta_1(q_2^*)\, ;\, q_2^* = \beta_2(q_1^*)$$

Hence, we can just solve the above system can get

$$q_1^* = q_2^* = a/3$$

Exercise.

Consider a two-firm industry selling homogeneous goods. Two firms are competing with quantities so that firm 1 chooses $q_1 \geq 0$ and firm 2 chooses $q_2 \geq 0$. The inverse demand is $P(Q) = a - Q$ where $a > 0$ and $Q = q_1 + q_2$ is total production of two firms. Suppose cost for firm $i = 1, 2$: $c_i(q_i) = cq_i$ where $a > c > 0$. Profit of firm 1 is $\pi_1 = (P(Q) - c_1)\, q_1$ and profit of firm 2 is $\pi_2 = (P(Q) - c_2)\, q_2$.

(a) Write down the normal-form for this game.

(b) What is the Nash equilibrium if $c = 0$? What is the Nash equilibrium?

(c) Draw best response correspondence to visualize Nash equilibrium.

(d) Apply iterated elimination of strictly dominated strategy to find Nash equilibirum.

(e) Consider asymmetric costs: $c_i(q_i) = c_i q_i$. What is Nash equilibirum if (i) $0 < c_i < a/2$? What if (ii) $c_1 < c_2 < a$ but $2c_2 > a + c_1$?

Suggested Solution.

(a) The game $G = (N, S, u)$ is:

$N = \{1, 2\}$

$S = S_1 \times S_2$ where $S_1 = S_2 = [0, a]$

$U = (u_1, u_2)$ where $u_1 = \pi_1$ and $u_2 = \pi_2$.

(b) For firm 1,

$$\max_{q_1} \left[a - (q_1 + q_2)\right] q_1$$

so that FOC is

$$q_1^* = \frac{a - q_2}{2}$$

We have to take corners into account. Obviously, producing at a is not optimal so we have

$$BR_1(q_2) = \max\left\{0, \frac{a - q_2}{2}\right\}$$

Similarly,

$$BR_2(q_1) = \max\left\{0, \frac{a - q_1}{2}\right\}$$

Solving the system, we have

$$q_1^* = q_2^* = \frac{a}{3}$$

Replacing a by $a-c$, then we have

$$q_1^* = q_2^* = \frac{a-c}{3}$$

c) Omitted.

d) For firm 1, any quantities $q_1 > (a-c)/2$ is strictly dominated. This is similar for firm 2. Hence, knowing this firm 1 will not choose any quantity $q_1 < (a-c)/4$ since they are best response to $q_2 > (a-c)/2$. Similar case for firm 2 again. Then firm 1 will not choose any quantity $q_1 > (3a-c)/8$. Hence, this converge to

$$q_1^* = q_2^* = \frac{a-c}{3}$$

e) The best responses are

$$\begin{aligned} BR_1(q_2) &= \max\left\{0, \frac{a-c_1-q_2}{2}\right\} \\ BR_2(q_1) &= \max\left\{0, \frac{a-c_2-q_1}{2}\right\} \end{aligned}$$

If we solve the equations, we can find the feasible candidates are

$$\begin{aligned} q_1^* &= \frac{a-2c_1+c_2}{3}; q_2^* = \frac{a-2c_2+c_2}{3} \\ q_1^* &= \frac{a-c_1}{2}; q_2^* = 0 \\ q_1^* &= 0; q_2^* = \frac{a-c_2}{2} \\ q_1^* &= 0; q_2^* = 0 \end{aligned}$$

Obviously, $q_1^* = 0; q_2^* = 0$ is not a NE. It is easy to check if $0 < c_i < a/2$, then the first one is the NE. And under (ii), second one is the NE.

Exercise.

Consider a two-firm industry selling homogeneous goods. All firms are competing with quantities so that firm i $(i = 1, 2)$ chooses $q_i \geq 0$. Inverse demand $P(Q) = a - Q$ where $a > 0$ and $Q = q_1 + q_2 + \cdots + q_n$. Cost function for firm $i = 1, 2, \ldots, n$ is $c_i(q_i) = cq_i$ where $a > c > 0$. Profit of firm $i = 1, 2, \ldots, n$ is $\pi_i(q_i, q_j) = [a - c - (q_i + q_j)]\, q_i$.

(a) Write down the normal-form for this game.

(b) Find the Nash equilibirum.

(c) What happen when n approach infinity?

Suggested Solution.

(a) $N = \{1, 2, \dots, n\}$

$S = S_1 \times S_2 \times \cdots \times S_n$ where $S_i = [0, a]$

$U = (u_1, u_2, \dots, u_n)$ where $u_i = \pi_i$.

(b) Similar to question 1, we have

$$BR_1(q_{-1}) = \max\left\{0, \frac{a - c - \sum_{j \neq 1} q_j}{2}\right\}$$

so that symmetric property implies

$$q_i^* = \frac{a - c}{n + 1}$$

(c) When n goes to infinity, each firm produces near zero quantity. Competitive equilibrium.

2.6 Existence of Pure NE

Please refer to the appendix for existence details. Here, we provide the main result:

Theorem. Debreu (1950): $\forall i \in N$, S_i is nonempty compact, convex and u_i is continuous and quasi-concave. Then, there exists Nash equilibrium in pure strategies.

2.7 Mixed Strategies

Example. Matching pennis Game

		2	
		H	T
1	H	-1,1	1,-1
	T	1,-1	-1,1

1. Rules: player 1 wins if pennies are different and player 2 wins if pennis are same

2. No pure strategy NE, this support the idea of randomization

Definition. A strategic game with Mixed Strategy $G = \left(N, (S_i, U_i)_{i \in N}\right)$.

1. For each $i \in N$, a mixed strategy $m_i : S_i \to [0,1]$ such that

$$\sum_{s_i \in S_i} m_i(s_i) = 1$$

2. The set of mixed strategy is $m_i \in M_i$ where $M_i = \Delta(S_i)$

3. The payoff function is vNM utility function: $U_i : \prod_{j \in N} M_j \to \mathbb{R}$ where

$$U_i(m) = \sum_{s \in S} \left[\prod_{j \in N} m_j(s_j) \right] u_i(s)$$

where $m = (m_i, m_{-i})$, $s = (s_i, s_{-i}) \in S$ and $S = \prod_{j \in N} S_j$.

Definition. A Nash equilibrium with mixed strategy is a strategy profile $m^* \in \prod_{j \in N} M_j$ such that

$$U_i\left(m_i^*, m_{-i}^*\right) \geq U_i\left(m_i, m_{-i}^*\right), \quad \forall m_i \in M_i$$

Example Matching pennis game revisited

		2	
		H	T
1	H	-1,1	1,-1
	T	1,-1	-1,1

1. $s_i = \{S, T\}$ so that $m_i : \{H, T\} \to [0,1]$ with the condition that $m_i(H) + m_i(T) = 1$.

2. Strategy profile $m = (m_1, m_2)$

3. Expected utility for player 1 is

$$\begin{aligned} U_1(m_1, m_2) &= m_1(H) m_2(H) u_1(H,H) + m_1(T) m_2(H) u_1(T,H) \\ &\quad + m_1(H) m_2(T) u_1(H,T) + m_1(T) m_2(T) u_1(T,T) \end{aligned}$$

4. Same for $U_2(m_1, m_2)$

5. NE is (m_1^*, m_2^*) such that

$$U_1(m_1^*, m_2^*) \geq U_1(m_1, m_2^*), \quad \forall m_1 \in M_1$$
$$U_2(m_1^*, m_2^*) \geq U_2(m_1^*, m_2), \quad \forall m_2 \in M_2$$

Example. Battle of Sexes revisited

		girl	
		opera	football
boy	opera	5,3	2,2
	football	2,2	3,5

1. Denote $m_1(\text{opera}) = r$ then $m_1(\text{football}) = 1 - r$

2. Denote $m_2(\text{opera}) = q$ then $m_1(\text{football}) = 1 - q$

3. Now player 1's expected utility would be

$$\begin{aligned} U_1 &= 2rq + (1-r)(1-q) \\ &= 3rq + 1 - r - q \end{aligned}$$

4. FOC implies that

$$\frac{\partial U_1}{\partial r} = 3q - 1 \gtreqless 0 \Leftrightarrow q \gtreqless \frac{1}{3}$$

so that

$$\beta_1(q) = \begin{cases} 0 & q < \frac{1}{3} \\ [0,1] & q = \frac{1}{3} \\ 1 & q > \frac{1}{3} \end{cases}$$

5. Similarly, for player 2's expected utility is

$$\begin{aligned} U_2 &= rq + 2(1-r)(1-q) \\ &= 2 - 2r - 2q + 3rq \end{aligned}$$

6. FOC implies that

$$\frac{\partial U_1}{\partial q} = 3r - 2 \gtreqless 0 \Leftrightarrow r \gtreqless \frac{2}{3}$$

so that

$$\beta_2(r) = \begin{cases} 0 & r < \frac{2}{3} \\ [0,1] & r = \frac{2}{3} \\ 1 & r > \frac{2}{3} \end{cases}$$

7. By drawing graphs, or by observation,

$$NE = \left\{ (\text{Football, Football}), (\text{Opera, Opera}), \left(r = \frac{2}{3}, q = \frac{1}{3} \right) \right\}$$

8. Note in the mixed strategy NE, the payoff is same for player no matter what is strategy so the main motive to randomize is to let the opponent to randomize.

2.7.1 Existence of mixed strategy NE

Every finite game has at least one Nash equilibrium (Nash 1950). See the proof in Appendix.

3 Dynamic Game of Complete Information

3.1 Contingent plan

In dynamic games, strategy is no longer action only but a contingent plan depends on the path of play, i.e. history of players' actions.

Example. Two symmetric firms compete in quantity and facing linear demand: $P(Q) = a - Q$ and $c_i(q_i) = cq$ where $a > 0$ and $c > 0$. If firm 1 is the first mover, this becomes a Stackerberg problem.

To solve the problem, we employ the backward induction technique,

$$\begin{aligned} q_2^* = \beta_2(q_1) &= \arg\max_{q_2} \pi_2 = \arg\max_{q_2} (a - q_1 - q_2 - c) q_2 \\ &= \frac{a - c - q_1}{2} \end{aligned}$$

and then firm 1 maximizes profit based on the best response of firm 2,

$$\begin{aligned} q_1^* = \beta_1(q_2) &= \arg\max_{q_1} (a - q_1 - \beta_2(q_1) - c) q_1 \\ &= \frac{a - c}{2} \end{aligned}$$

so that

$$q_2^* = \frac{a - c}{4}$$

3.2 Extensive Form Game

Extensive Form Game: $G = \{N, T, I, n, A, u, P\}$
Set of Players: $N = \{1, 2, \ldots, N\}$
Set of Nodes: T where $Z \subset T$ is terminal nodes and all $t \in T \backslash Z$ are decision nodes (non-terminal node).
Player moves at node t: $i\,(t)$ where $i : T \to N$
Set of Action at node t: $A(t)$
Successor node: $n(t, a)$ where $n : (T \backslash Z) \times A \to T$
Payoff functions: $u_i : Z \to \mathbb{R}$
Information Set $P(t)$: set of nodes player $i(t)$ knows it is possible; it is a partition:

$$t' \in P(t) \text{ implies } i(t') = i(t), A(t') = A(t),\ P(t') = P(t)$$

Definition. A subgame is a part of the game that starts with a decision node and contains all the sucessor nodes without cutting any information sets.
Definition. A subgame perfect equilibrium is a Nash equilibrium for every subgame game.
Motivating Example. Empty threat of firm entry

1. Two players: incumbent and potential entrant.
2. Zero payoff for both if no entry. If there is an entry, incumbent can choose to fight or accommodate. Both have loss of 5 if there is fight. If the incumbent accommodates, the incumbent loses 1 and the entrant earns 1. Draw a diagram here.
3. Though no entry and fighting is a NE, but it is not a credible threat as entrant will actually accommodate if there is a entry.
4. Selten (1965) suggests the idea of subgame perfect Nash equilibrium which is essentially the idea of combination of backward induction and Nash equilibrium.
5. Subgam-perfect means we require Nah equilibirum in every subgame.

3.3 Repeated game

Stage game: $G = \left(N, (A_i, u_i)_{i \in N}\right)$ where A_i is action set and $u_i : A \to \mathbb{R}$ where $A = \prod_{j \in N} A_j$

Number of stage: $T+1$ times: $0,1,2,\ldots,T$
Path $\mathbf{a}=\{a(t)\}_{t=0}^{T}$
Payoff: $U_i(\mathbf{a})=\sum_{t=0}^{T}u_i(a(t))$
t-history: $h(t)=(a(0),a(1),\ldots,a(t-1))\in H(t)=A^t$
strategy σ_i at period t: $\sigma_i(t):H(t)\to A_i$ or $A^t\to A_i$
strategy of the game σ_i: $\sigma_i=(\sigma_i(t))_{t=0}^{T}:\bigcup_{t=0}^{T}A^t\to A_i$
strategy profile: $\sigma=(\sigma_i)_{i\in N}$
Path generated from σ: $\mathbf{a}(\sigma)$
Path generated from σ after $h(t)$: $\mathbf{a}(\sigma,h(t))$
Nash equilibrium:

$$\forall i\in N, U_i(\mathbf{a}(\sigma))\geq U_i(\mathbf{a}(\sigma_{-i},\sigma_i')) \quad \forall\sigma_i'$$

Subgame perfect Nash equilibrium:

$$\forall i\in N,\forall t,\forall h(t)\in A^t, U_i(\mathbf{a}(\sigma,h(t)))\geq U_i(\mathbf{a}(\sigma_{-i},\sigma_i',h(t))) \quad \forall\sigma_i'$$

3.3.1 Finitely Repeated Game

Theorem. Suppose that the game G is repeated $T+1$ times. If G has a unique Nash equilibrium a^*, then the repeated game has a unique SPNE of which the equilibrium path is

$$\mathbf{a}(\sigma)=(a^*,\ldots,a^*)$$

When it comes to non-uniqueness of stage NE then non-NE outcome might be supportable in the early stages.
Example:

		2		
		B	C	D
	B	3,3	0,0	0,0
1	C	0,0	4,4	0,5
	D	0,0	5,0	1,1

where $\mathbf{a}=((C,C),(D,D))$ can be supported as SPNE

3.3.2 Infinitely repeated game

To make payoff comparable, we allow player discount future payoff to make it finite.

$$U_i(\mathbf{a}) = \sum_{t=0}^{\infty} \delta^t u_i(a(t))$$

		2	
		C	D
1	C	4,4	0,5
	D	5,0	1,1

To support $\mathbf{a} = \{(C, C), (C, C), \ldots\}$, the threat to be is to play $\{D, D\}$ forever. The requirement would be

$$\frac{4}{1-\delta} \geq 5 + \delta \frac{1}{1-\delta}$$
$$\delta \geq \frac{1}{4}$$

Friedman's **folk theorem** (1971): When $\delta \to 1$, any outcome giving payoff greater than the worst NE payoff can be supported.

4 Static Game of Incomplete Information

4.1 Bayesian game

Game is described by $\left(N, (A_i, T_i, P_i, u_i)_{i \in N}\right)$ where N is player set, A_i is action set, T_i is set of types, $P_i\left(t_{-i} \mid t_i\right)$ is belief of type i of player i, strategy is $s_i : T_i \to A_i$ and $u_i\left(a_1, \ldots, a_n; t_1, \ldots, t_n\right)$ is payoff function.
This is an extensive form game where there is a player called nature moves first and this player is not strategic.
To simplify the case, we adopt the common belief assumption

$$p_i = p\left(t\right) = p\left(t_1, t_2, \ldots, t_n\right) \text{ for all } i \in N$$

Now, we have

$$p_i\left(t_{-i} \mid t_i\right) = \frac{p\left(t_{-i}, t_i\right)}{\sum_{t_i' \in T_i} p\left(t_{-i}', t_i\right)}$$

Note the common belief does not implies whether belief is independent or dependent on types. Example:

$$\begin{array}{cccc} & & 2 & \\ & & W_2 & S_2 \\ 1 & W_1 & 1/15 & 4/15 \\ & S_1 & 2/15 & 8/15 \end{array} \quad \text{and} \quad \begin{array}{l} P_1\left(W_2 \mid W_1\right) = \frac{1/15}{5/15} = \frac{1}{5} \\ P_1\left(W_2 \mid S_1\right) = \frac{2/15}{10/15} = \frac{1}{5} \end{array}$$

and

$$\begin{array}{cccc} & & 2 & \\ & & W_2 & S_2 \\ 1 & W_1 & 1/15 & 4/15 \\ & S_1 & 8/15 & 2/15 \end{array} \quad \text{and} \quad \begin{array}{l} P_1\left(W_2 \mid W_1\right) = \frac{1/15}{5/15} = \frac{1}{5} \\ P_1\left(W_2 \mid S_1\right) = \frac{8/15}{10/15} = \frac{4}{5} \end{array}$$

4.2 Bayesian Nash equilibrium

Definition. Bayesian NE is $s^* = \left(s_1^*, \ldots, s_n^*\right)$ such that $\forall i, \forall t_i$

$$s_i^*\left(t_i\right) = \arg\max_{a_i \in A_i} \sum_{t_{-i} \in T_{-i}} u_i\left(s^*\left(t_1\right), \ldots, a_i, \ldots s_n^*\left(t_n\right); t\right) p_i\left(t_{-i} \mid t_i\right)$$

where $s_i^* : T_i \to A_i$ in fact is best response to s_{-i}^*.
To rewrite it in a compact manner, we have $s_i^* \in \beta_i\left(s_{-i}^*\right)$

4.2.1 Application: Cournot Game with incomplete information

firm 1's marginal cost: c

firm 2's marginal cost: $c_H > c_L$

firm 1's belief: $p(c_2 = c_H) = \theta$ and $p(c_2 = c_L) = 1 - \theta$

firm 1's belief is common knowledge

Now $q_2 : \{c_H, c_L\} \to A$, we have firm 2's maximization:

If c_H, then we have $\pi_2 = [(a - q_1^* - q_2) - c_H] q_2$ so that

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2}$$

If c_L, then we have $\pi_2 = [(a - q_1^* - q_2) - c_L] q_2$ so that

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}$$

Firm 1's maximization would be

$$\begin{aligned} q_1^* &= \arg\max_{q_1} \theta [a - q_1 - q_2^*(c_H) - c] q_1 + (1 - \theta) [a - q_1 - q_2^*(c_L) - c] q_1 \\ &= \frac{a - c - [\theta q_2^*(c_H) + (1 - \theta) q_2^*(c_L)]}{2} \\ &= \frac{a - 2c + \theta c_H + (1 - \theta) c_L}{3} \end{aligned}$$

so that

$$\begin{aligned} q_2^*(c_H) &= \frac{a - 2c_H + c}{3} + \frac{(1 - \theta)(c_H - c_L)}{6} \\ q_2^*(c_L) &= \frac{a - 2c_L + c}{3} - \frac{\theta (c_H - c_L)}{6} \end{aligned}$$

5 Dynamic Game of Incomplete Information

5.1 Perfect Bayesian Equilibrium

A PBE is a pair of strategy profile and system of belief for every information set.
Two requirements:

1. sequential rationality: given the belief at each information set, expected utility is maximized.

2. Bayesian updating: on equilibrium path, belief are revised accoding to Bayes theorem

5.1.1 Signalling Game

Two players: sender S and receiver R
Stage 1: The sender S moves first by sending a message.
Stage 2: The receiver R gets the message and then choose an action
The game $G = \{\{S, R\}, T, P, M, A, u_s, u_r\}$ where

1. M: S's action

2. A: R's action

3. T: S's type set

4. P: R's prior belief

5. u_s: $u_s(m_j, a_k, t_i)$: S's payoff function

6. u_r: $u_r(m_j, a_k, t_i)$: R's payoff function

5.1.2 Perfect Bayesian Equilibrium

Requirement 1: μ is a system of belief

$$\mu(t_i \mid m_j) \text{ s.t. } \sum_{t_i \in T} \mu(t_i \mid m_j) = 1$$

Requirement 2: sequential rationalities
S's sequential rationality:

For each type i,

$$m^* (t_i) \in \arg \max_{m_j \in M} u_s (m_j, a (m_j), t_i)$$

R's sequential rationality:

$$a^* (m_j) \in \arg \max_{a_k \in A} \sum_{t \in T} u_r (m_j, a_k, t_i) \mu (t_i \mid m)$$

Requirement 3: Bayesian updating
For each $m_j \in M$, if $m^* (t_i) = m_j$ for some t_i,

$$\mu (t_i \mid m_j) = \frac{p (t_i)}{\sum_{t'_i \in T_j} p (t'_i)}$$

where $T_j = \{t'_i \in T, m^* (t'_i) = m_j\}$

A Appendix. Advanced Material

A.1 Static Game of Complete Information

A.1.1 Iterated elimination and Nash equilibrium:

Proposition. Suppose G is a finite game, If $s^* \in \prod_{j \in N} S_j$ is the unique survivor of i.e.s.d.s., then s^* is a Nash equilibrium.
Proof: See Appendix.

1. Suppose not. s^* is the unique survivor of i.e.s.d.s. but s^* is not Nash equilibrium

2. This implies that $\exists i \in N$, $\exists s_i \in S_i$ such that

$$u_i \left(s_i^*, s_{-i}^*\right) < u_i \left(s_i, s_{-i}^*\right)$$

3. However, given s^* being the unique survivor, s_i is eliminated at some stage so that $\exists s'_i \in S_i$ such that

$$u_i (s_i, s_{-i}) < u_i (s'_i, s_{-i}), \quad \forall s_{-i} \in \tilde{S}_{-i}$$

4. Yet, s^*_{-i} is not eliminated in every stage so that

$$u_i\left(s_i, s^*_{-i}\right) < u_i\left(s'_i, s^*_{-i}\right)$$

5. However, s'_i is also eliminated at some stage so that there exists s''_i such that

$$u_i\left(s'_i, s^*_{-i}\right) < u_i\left(s''_i, s^*_{-i}\right)$$

6. Hence, by transitivity, we have

$$u_i\left(s^*_i, s^*_{-i}\right) < u_i\left(s_i, s^*_{-i}\right) < u_i\left(s'_i, s^*_{-i}\right) < u_i\left(s''_i, s^*_{-i}\right) < \cdots$$

7. And given finite iteration and the fact that s^* is the last survivor, we have

$$u_i\left(s_i, s^*_{-i}\right) < u_i\left(s'_i, s^*_{-i}\right) < u_i\left(s''_i, s^*_{-i}\right) < \cdots < u_i\left(s^*_i, s^*_{-i}\right)$$

8. Hence, contradiction.

Proposition. If s^* is a Nash equilibrium, then s^* survives i.e.s.d.s.
Proof:

1. Suppose not. s^* is a Nash equilibrium but s^* does not survive i.e.s.d.s.

2. This implies at some stage, for some $i \in N$, s^*_i is eliminated due to strictly dominated: that is, $\exists i \in N$, $\exists s'_i \in \tilde{S}_i$ such that

$$u_i\left(s'_i, s_{-i}\right) > u_i\left(s^*_i, s_{-i}\right), \quad \forall s_{-i} \in \tilde{S}_{-i}$$

3. Choose the first i that satisfies the above (first i because the elimination is done in the order i), we would have $s^*_{-i} \in \tilde{S}_i$

$$u_i\left(s'_i, s^*_{-i}\right) > u_i\left(s^*_i, s^*_{-i}\right)$$

   and hence a contradiction to the definition of Nash equilibrium.

A.1.2 Existence of Nash Equilibrium

Mathematics

1. sequence $S \subseteq \mathbb{R}$: $\{s_k\}_{k=0}^{\infty}$, $s_k \in S$, $\{s_1, s_2, s_k, \ldots\}$

2. convergent sequence: $\forall \varepsilon > 0$, $\exists k(\varepsilon) \in \mathbb{N}$ s.t. $\forall k, k' \geq k(\varepsilon)$, $\|s_k - s'_k\| < \varepsilon$

3. closed set S: any converging sequence has limit in S

   open set: complement of closed set

4. Continuous function $f : S \to T \subseteq \mathbb{R}$ is continuous if for all convergent sequence $\{s_k\}_{k=0}^{\infty} \to s$, then $\{f(s_k)\}_{k=0}^{\infty} \to f(s)$.

5. Quasi-concave function $f : S \to \mathbb{R}$, $\forall s, s' \in \mathbb{R}$, $\forall \lambda \in [0,1]$, $f(\lambda s + (1-\lambda) s') \geq \min\{f(s), f(s')\}$

6. Nonemptiness of S: $S \neq \{\varnothing\}$

7. Boundedness of S: $\forall s_i \in S$, $\exists r \in \mathbb{R}_{++}$ such that $\|s_i\| \leq r$

8. Compactness of S: closed and bounded (doesn't have to be connected)

   Any sequence in compact set has a converging subsequence

9. Convex set X: any linear combination of element in the each is inside the set

10. Correspondence $\beta : S \to `T$, $\forall s \in S$, $\beta(s) \subset T$

    β is nonempty-valued if $\beta(s) \neq \varnothing$ $\forall s \in S$

    β is convex-valued if $\beta(s)$ is convex $\forall s \in S$

11. Upper-hemi continuous (UHC):

    $\forall \{s_k\}_{k=0}^{\infty} \to S$, $\forall \{b_k\}_{k=0}^{\infty}$ such that $b_k \in \beta(s_k)$, $\forall$convergent subsequence $\{b'_k\}_{k=0}^{\infty}$ of $\{b_k\}_{k=0}^{\infty}$, then $b \in \beta(s)$.

NE is a fixed point:

1. Game $G = \left(N, (S_i, u_i)_{i \in N}\right)$

2. Best response correspondence: $\beta_i(s_{-i}) = \{s_i \in S_i \mid u(s_i, s_{-i}) \geq u(s'_i, s_{-i}), \forall s'_i \in S_i\}$

3. Nash equilibrium s^* is a fixed point: $s^* \in S$ such that $s^* \in \beta(s^*)$ where $\beta = \prod_{i=1}^{n} \beta_i$ and $\beta_i : S_{-i} \to S_i$

Kakutani fixed point theorem:

1. $S \subseteq \mathbb{R}^N$ and $\phi : S \to S$

2. S is nonempty, compact and convex

3. ϕ is UHC, non-empty valued and convex

4. Then $\exists s^* \in S$ such that $s^* \in \phi(s^*)$

Debreu (1958): Existence of pure NE Statement: $\forall i \in N$, s_i is nonempty compact, convex and u_i is continuous and quasi-concave. Then, there exists Nash equilibrium in pure strategies
Proof:

1. The condition for S is assumed.

2. Impose condition on u_i such that β_i satisfies the condition $\beta_i : S_{-i} \to S_i$

3. Non-emptiness of β_i is ensured by Weierstress's theorem (every continuous function in a finite-dimensional non-empty compact Euclidean space has a solution to a maximization problem) : $u_i(s_i, s_{-i})$ is continuous in s_i and S_i is compact then there exists a maximum and hence β_i is non-empty.

4. Convexed-valuedness of β_i: by quasi-concavity in u_i (by definition, a function is quasiconcave on a convex set if every upper level set of this function is convex)

5. UHC of β_i is ensured by Berge's maximum theorem: $u_i(s_i, s_{-i})$ is continuous in $S_i \times S_{-i}$, then β_i is UHC

6. Hence, β is UHC, non-empty valued and convex. Apply Kakutani fixed point theorem.

Existence of NE in Cournot Model. Theorem: There exists a Cournot equilibrium if

1. inverse demand function $P(Q)$ satisfies $\exists \bar{Q}$ s.t. $\forall Q \geq \bar{Q}$, $P(Q) = 0$,

2. $P(Q)$ is continuous and decreasing. (Rmk: this implies $\lim_{x \to 0} P(x) < \infty$),

3. $c_i(q_i)$ is continuous and increasing in q_i

4. $c_i(q_i)$ is convex in q_i

5. $P(Q)$ is concave.

Proof:

1. By construction, S is non-empty and convex.

2. It is closed and bounded by assumption 1.

3. From assumption 3, $\pi_i(q_i, q_{-i}) = P\left(q_i + \sum_{j \neq i} q_j\right) q_i - c_i(q_i)$ is continuous in q_i so that by Weirestess theorem, the any continuous function has maximum in a compact set, hence φ is non-empty.

4. Note that from assumption 3, π_i is also continuous in q_j, by Berge's maximum theorem, φ is upper hemi-continuous.

5. Since in single variable function, quasi-concavity is equivalent to single peakedness, then from assumption 4 and 5, we know φ is convex valued by quasi-concavity of π_i.

Theorem: With the above 5 assumptions, there exists a Cournot equilibrium:
Proof:

1. By construction, S is non-empty and convex.

2. It is closed and bounded by assumption 1.

3. From assumption 3, $\pi_i(q_i, q_{-i}) = P\left(q_i + \sum_{j \neq i} q_j\right) q_i - c_i(q_i)$ is continuous in q_i so that by Weirestess theorem, the any continuous function has maximum in a compact set, hence φ is non-empty.

4. Note that from assumption 3, π_i is also continuous in q_j, by Berge's maximum theorem, φ is upper hemi-continuous.

5. Since in single variable function, quasi-concavity is equivalent to single peakedness, then from assumption 4 and 5, we know φ is convex valued by quasi-concavity of π_i.

Nash (1950): Existence of mixed strategy NE Every finite game has at least one Nash equilibrium.

Definiition. Finite game: finite number of players and finite number of pure strategies.

1. Finite Game: $G = \left(N, (S_i, u_i)_{i\in N}\right)$ where u_i is vNM utility
2. Mixed strategy: $m_i : S_i \to [0,1]$ with $\sum_{s_i \in S_i} m(s_i) = 1$ such that M_i is convex and compact
3. The utility:

$$\begin{aligned} U_i(m_i, m_{-i}) &= \sum_{s\in S}\left[\prod_{j\in N} m_j(s_j)\right] u_i(s) \\ &= \sum_{s\in S} m_i(s_i) \sum_{s_{-i}\in S_{-i}}\left[\prod_{j\neq i} m_j(s_j)\right] u_i(s_i, s_{-i}) \\ &= \sum_{s\in S} m_i(s_i)\, \tilde{U}_i(s_i, m_{-i}) \end{aligned}$$

   hence utility U_i is continuous and convex (due to $U_i(m_i, m_{-i})$ is linear). Apply Kakutani fixed point theorem.

A.1.3 Application: Model of Sales (Varian 1980)

Setup:

1. N firms
2. a continnum of consumers
3. type of consumer: informed I and uninformed U. (uninformed are locked into particular firms; informed will find the lowest price)
4. population is 1: $I + U = 1$
5. Willingness to pay v is common across all consumers
6. Marginal cost of production is zero ($MC = 0$)
7. Each firm i announces price $p_i \geq 0$ simultaneously as a pure strategy.

8. Firm(s) with highest price will have (to split) the informed consumers. Uninformed consumers are evenly shared by all firms. Denote k to be number of firms charging the lowerest price. Hence, profit function would be

$$\pi_i(p_i, p_{-i}) = \begin{cases} p_i \frac{U}{N} & \text{if } p_i > \min_j p_j \\ p_i \left(\frac{U}{N} + \frac{I}{k}\right) & \text{if } p_i = \min p_j \text{ and } |\{j : p_i = p_j\}| = k \end{cases}$$

Claim. There is no pure strategy Nash equilibrium in this game.
Proof.
Case 1: $k > 1$.

1. If $v > \min p_j > 0$, then one of k firms charge $p_j - \varepsilon$, get the whole market I.
2. If $v > \min p_j = 0$, then one firm can charge v and have profit.
3. If $v = \min p_j$, then firm can charge $v - \varepsilon$, grab the whole market.

Case 2: $k = 1$
Suppose i is the minimal.
We have $p_i = \min p_j$ and $p_j = v$ for $j \neq i$.
This implies $\pi_i(p_i, p_{-i}) = \pi_j(v, p_{-j})$ so that firm i wants to charge higher until $v - \varepsilon$.
Hence, there cannot be any pure strategy equilibrium.
Exercise. Location choice. Consumers are uniformly located in a linear city of length 1. Each consumer only need to buy one good from a firm. Consumer needs to incur transportation cost of buying from a firm that is proportional to the distance from that firm. Prices are fixed by government. Hence, consumers go to the nearest firm. If more than one firms are in the same location, they share the consumer equally. There are two firms choosing their locations simultaneously
(i) Formally write down the normal-form game.
(ii) Show that there is unique pure-strategy Nash equilibirum that both firms are located in the middle of the city.
(iii) Show that when there are three firms are choosing their locations, there is no-pure strategy Nash equilibrium.[1]

[1] For characterization of mixed strategy, see A. Shaked(1982) "Existence and Computation of Mixed Strategy Nash Equilibrium for 3-Firms Location Problem, " The Journal of Industrial Economics, Vol. 31, No. 1/2, Symposium on Spatial Competition and the Theory of Differentiated Markets (Sep. - Dec., 1982), pp. 93-96.

Mixed strategy Nash equilibrium

Focus on symmetric mixed strategy equilibrium:

Denote $F(p)$ be the cdf of strategy p. Let $\bar{p}(F)$ and $\underline{p}(F)$ be two endpoints of support of F:

$$\bar{p}(F) = \inf\{p \mid F(p) = 1\},$$
$$\underline{p}(F) = \sup\{p \mid F(p) = 0\}$$

Theorem: There is unique symmetric equilibirum such that

$$(i) : \bar{p}(F) = v$$
$$(ii) : \underline{p}(F)\left(\frac{U}{N} + I\right) = v\frac{U}{N}$$
$$(iii) : p\left[\frac{U}{N} + (1 - F(p))^{n-1} I\right] = v\frac{U}{N} \text{ for all } p \in \left[\underline{p}(F), \bar{p}(F)\right]$$

(i) $\bar{p}(F) < v$ implies there is expected non-zero profit charging in between $\bar{p}(F)$ and v.

(iii) With (i), there is no spike in F, we have (iii). (No firm can charge v and happier)

And with no gaps, we have (ii). (Charging at the lowest price same as charging v)

Intuition: some firms offer sales and some firms offer no discount. They have to randomize to avoid being detected informed consumer.

A.2 Dynamic Games of Complete Information

A.2.1 Application: Stackelberg-Cournot Model

Stage 1: Firm 1 makes commitment in capital investment K_1

Stage 2: Firm 2, after observing this, decides to enter or not. If enter, pay entry cost f and investement capital K_2.

Both firms have zero marginal cost.

The payoff for firm 1 would be $\pi_1(K_1, K_2) = K_1(1 - K_1 - K_2)$

The payoff for firm 2 would be

$$\pi_2(K_1, K_2) = \begin{cases} K_2(1 - K_1 - K_2) - f & \text{if } K_2 > 0 \\ 0 & \text{if } K_2 = 0 \end{cases}$$

By backward induction, suppose there is entry. Then

$$\beta_2(K_1) = \frac{1-K_1}{2}$$
$$\pi_2(K_1, \beta_2(K_1)) = \left(1 - K_1 - \frac{1-K_1}{2}\right)\frac{1-K_1}{2} - f = \left(\frac{1-K_1}{2}\right)^2 - f$$
$$\pi_1(K_1, \beta_2(K_1)) = \left(1 - K_1 - \frac{1-K_1}{2}\right)K_1$$

So that solving gives

$$\pi_1^* = 1/8 \text{ and } \pi_2^* = 1/16$$

In SPNE, to prevent entry

$$\max_{K_2}\left[K_2(1-K_1-K_2) - f\right] = 0$$

or

$$\left(\frac{1-K_1}{2}\right)^2 = f \Rightarrow K_1 = 1 - 2\sqrt{f}$$

Hence firm 1's payoff is

$$\begin{aligned}\pi_1(K_1, \beta_2(K_1)) &= \left(1 - \left(1 - 2\sqrt{f}\right)\right)\left(1 - 2\sqrt{f}\right) \\ &= 2\sqrt{f}\left(1 - 2\sqrt{f}\right)\end{aligned}$$

So that entry is preferred if

$$2\sqrt{f}\left(1 - 2\sqrt{f}\right) \geq 1/8$$

Hence, there is no entry if $0.00536 < f < 0.182$.
However, if commiment cost is non-zero but small, then it might not be possible to achieve this equilibirum as here commitment is free.

A.2.2 Application: Strategic trade model

Two countries, two firms and two consumers.
Cournot: Brander and Spencer (1985)
Stage 1: government 1 and 2 simultaneous decides their tax (subsidy) by maximization of social welfare

Stage 2: firm 1 and firm 2 simultaneous decide their production and export by maximization of profit

1. Setup:

   Inverse demand function: $P = 1 - Q$

   Marginal cost: $c = 0$

   Profit function of firm in country i: $\pi_i(q_i, q_j; s_i) = (1 - q_i - q_j) q_i + s_i q_i$

   Government i: $G_i(q_i, q_j; s_i) = \pi_i(q_i, q_j; s_i) - s_i q_i$

2. Backward induction: FOC would be

   $$\frac{\partial \pi_i}{\partial q_i} = 1 - 2q_i - q_j + s_i = 0$$

   Best response of firm i:

   $$\beta_i(q_j; s_i) = \frac{1 - q_j + s_i}{2}$$

   and hence, we have

   $$q_i^* = \frac{1 - s_j + 2s_i}{3} \text{ and } q_j^* = \frac{1 - s_i + 2s_j}{3}$$

3. First stage game becomes

   $$\begin{aligned} G_i(q_i, q_j; s_i) &= \left(1 - \frac{2 + s_j + s_i}{3}\right) q_i - s_i q_i \\ &= \left(\frac{1 - s_j - 4s_i}{3}\right) \frac{1 - s_j + 2s_i}{3} \end{aligned}$$

   where FOC becomes

   $$\frac{\partial G}{\partial s_i} = \frac{-4 + 4s_j + 2 - 2s_j - 16s_i}{9} = 0$$

   or

   $$s_i = \frac{1 + s_j}{16}$$

   so that $s_i = s_j = 1/15$. Government would want to subsidize.

Betrand: Eaton and Grossman (1986)

A.2.3 Collusion in Discret Cournot Game: Dilip Abreu (1988) Penal code

Consisder the follwoing:

		2		
		L	M	H
	L	10,10	3,15	0,7
1	M	15,3	7,7	-4,5
	H	7,0	5,-4	-15,-15

which has (M, M) as NE.
Using $\mathbf{a}^* = \{(M, M), (M, M), \ldots\}$, we could support $\mathbf{a}^0 = \{(L, L), (L, L), \ldots\}$ if

$$\frac{10}{1-\delta} \geq 15 + \delta \frac{7}{1-\delta}$$
$$\delta \geq \frac{5}{8}$$

However, this is not the lower bound.
Now suppose the penal code to be

$$\mathbf{a}^1 = \{(M, H), (L, M), (L, M) \ldots\}$$
$$\mathbf{a}^2 = \{(H, M), (M, L), (M, L) \ldots\}$$

Penal code means If a player deviates from the designated path, he would be punished according to his punishment. On the punishment path, if any player deviates from the designated path, he would be punished according to his punishment.
Exercise. Show that $\delta \geq 4/7$ is the lower bound.
To check whether such a path is SPNE, one has to resort to one-deviation principle to check whether each player would have incentive to deviate once at each the equilibrium path of play.

A.2.4 Application: Price War (Rotemberg and Saloner,1986 AER)

Assumption:

1. Two firms are in Bertrand competition every period.
2. Each period demand $D(t)$ for goods can be Q_L or Q_H.

3. Q_H or Q_L is observable in the beginning of each period.

4. The probability being Q_L and Q_H are $1-w$ and w

5. Demand across time is independent: $D(t)$ and $D(t+h)$ are indepedent for any $h \neq 0$.

6. Consumers willingness to pay is 1.

7. There are two firms in the market. They compete in Bertrand competition:

$$\pi_i\left(p_i\left(t\right), p_j\left(t\right); Q\left(t\right)\right) = \begin{cases} P_i\left(t\right) Q\left(t\right) & \text{if } P_i\left(t\right) < P_j\left(t\right) \\ \frac{P_i(t)Q(t)}{2} & \text{if } P_i\left(t\right) = P_j\left(t\right) \\ 0 & \text{if } P_i\left(t\right) > P_j\left(t\right) \end{cases}$$

8. History at time t would be $h\left(t\right) = \left\{\left(Q\left(t'\right), P_i\left(t'\right), P_j\left(t'\right)\right)_{t'=0}^{t}\right\}$

9. Strategy at time t owuld be $\sigma_i\left(t\right) : H\left(t\right) \times \{Q_H, Q_L\} \to \mathbb{R}_+$

Consider equilibrium with trigger strategy:
Claim: There exists a symmetric trigger strategy equilibrium that maximize the total payoff where

$$P_i\left(t\right) = P_j\left(t\right) = P\left(Q\left(t\right)\right)$$

such that

$$\begin{cases} P\left(Q_L\right) = P\left(Q_H\right) = 1 & \text{if } \delta > \frac{Q_H}{(1+w)Q_H + (1-w)Q_L} \\ P\left(Q_L\right) = 1; P\left(Q_H\right) = \dfrac{\delta\left(1-w\right)Q_L}{Q_H\left(1-\delta\left(1+w\right)\right)} & \text{if } \frac{1}{2} \leq \delta \leq \frac{Q_H}{(1+w)Q_H + (1-w)Q_L} \\ P\left(Q_L\right) = P\left(Q_H\right) = 0 & \text{if } \delta < 1/2 \end{cases}$$

Under trigger strategy, after deviation, continuation value is zero.
Under cutting is $P\left(Q_H\right)$ is unprofitable iff

$$\frac{P\left(Q_H\right)Q_H}{2} + \delta\frac{wP\left(Q_H\right)Q_H + \left(1-w\right)P\left(Q_L\right)Q_L}{2\left(1-\delta\right)} \geq P\left(Q_H\right)Q_H$$
$$\frac{P\left(Q_L\right)Q_L}{2} + \delta\frac{wP\left(Q_H\right)Q_H + \left(1-w\right)P\left(Q_L\right)Q_L}{2\left(1-\delta\right)} \geq P\left(Q_L\right)Q_L$$

which implies

$$\delta \geq \frac{P\left(Q_H\right) Q_H}{(1+w) P\left(Q_H\right) Q_H+(1-w) P\left(Q_L\right) Q_L}$$
$$\delta \geq \frac{P\left(Q_L\right) Q_L}{(1+w) P\left(Q_H\right) Q_H+(1-w) P\left(Q_L\right) Q_L}$$

Exercise. Check the above inequalities to find the thresolds of δ.
Note that in this model,

$$P\left(Q_H\right) \leq P\left(Q_L\right)$$

which implies lower price during boom!

A.3 Static Games of Incomplete Inforomation

A.3.1 Application: Contract Theory

Mechanism Design A social choice problem.
There are n agent in the society.
Each agent i has type θ_i.
Social optimal allocation (despite Arrow's impossbility theorem) would be

$$y(\theta)=\left(y_1(\theta), \ldots, y_n(\theta)\right)$$

where $\theta=\left(\theta_1, \ldots, \theta_n\right)$
Hard to implement $y(\theta)$ because each agent i has incentive to maniupulate θ_i to improve allocation.
A mechanism is $(y, M_1, \ldots, M_n)$ such that each agent i submits message $m_i \in M_i$ and $y(m)$ is the allocation rule. The informaiton set of agent i is I_i which may be include some subset of θ_{-i}.
An equilibrium is such that $y^*(I_1, \ldots, I_n)=y(m_1^*(I_1), \ldots, m_n^*(I_n))$.

Adverse Selection Problem Principle-agent model: $n=1$.
The problem becomes (y, M) and $I=\theta$ so the equilibirum is

$$m^*(\theta) \in \underset{m \in M}{\operatorname{argmax}}\, u(y(m), \theta)$$

where the allocation is $y^*(\theta)=y(m^*(\theta))$.

A direct mechanism: report directly the type (message space is the type space) $M = \Theta$.
A truthful mechanism: report its own type

Revaluation Principle Theorem. If $y^*(\theta)$ is implmented through some mechanism, then it can also be implemented through a direct truthfl mechanism where the agent reveals his information.
Proof.

1. Let (y, M) be the mechanism implement y^* and $m^*(\theta)$ be the equilibirum mesage. Hence $y^* = y(m^*(\theta))$.

2. Suppose a direct mechanism (y^*, Θ) but it is not truthful. This implies there exists $\theta' \neq \theta$ such that
$$u(y^*(\theta), \theta) < u(y^*(\theta'), \theta).$$

3. However, this implies
$$u(y(m^*(\theta)), \theta) < u(y(m^*(\theta')), \theta)$$
which contradicts our given condition. QED

General Revelation Principle In words: For any Bayesian game and any Bayesian Nash equilibirum of the game, we can construct a new type-reporting Bayesian game in which truth-telling is weakly dominant strategy for each player.
More formally, suppose we have a Bayesian game $G = \left(N, (A_i, T_i, P_i, u_i)_{i \in N}\right)$. A BNE $s^* = (s_1^*, s_2^*, \ldots, s_n^*)$ is defined as

$$\forall t_i \in T_i,\ s_i^*(t_i) = \arg\max_{a_i \in A_i} \textstyle\sum_{t=i \in T_{-i}} u\left(s_{-i}^*, a_i, t\right) p_i\left(t_{-i} \mid t_i\right)$$

A transformed game called type-reportting game is that

$$G' = \left(N, (T_i, T_i, P_i, v_i)_{i \in N}\right)$$

where

$$\begin{aligned} v_i &= v_i\left(\tau_1, \ldots, \tau, t_1, \ldots, t_n\right) \\ &= u_i\left(s_i^*\left(\tau_i\right), s_{-i}^*\left(\tau_{-i}\right), t\right) \end{aligned}$$

Revelation Principle means that truth telling strategy σ^* is a BNE where $\sigma_i^*(t_i) = t_i$ for all i and for all t_i

A.3.2 Applicaton: Vertical Differentiation/Price Discrimination

Discrete Version Monopolist problem: (Mussa and Rosen 1998 JET)

1. Different qualities can be provided by a monopolist
2. multiple types of consumers
3. each consumer buys at most 1 unit
4. monopolist cannot observe consumer's type
5. monopolist offers a price schedule ever different quality goods

Setup:

1. Two consumers types: $T = \left\{\theta^H, \theta^L\right\}$ where $\theta^H > \theta^L > 0$
2. Belief: $p\left(\theta^H\right) = \alpha$ and $p\left(\theta^L\right) = 1 - \alpha$
3. Quantity: $q \in [0, \infty)$
4. marginal cost of producing type q: $c(q) > 0$, $c'(q) > 0$, $c''(q) > 0$ with $c(0) = 0$
5. Firm offers price for every quantity: $p(q)$
6. Consumer with type θ^t has utility $\theta^t q - p(q)$

A first-best solution: (First-order price discrimination)

1. Suppose θ^t is observable.
2. Firm just needs to provide two different price-quantity pairs $(p(\theta^t), \theta^t)$ for $t \in T$
3. Now the firm's problem becomes

$$\max_{p^t} \left[p\left(q\left(\theta^t\right)\right) - c\left(q\left(\theta^t\right)\right)\right]$$

   such that

$$\theta^t q(\theta^t) - p\left(q\left(\theta^t\right)\right) \geq 0$$

4. The FOCs are

$$
\begin{aligned}
c'(q^*(\theta^t)) &= \theta^t \\
p\left(q^*\left(\theta^t\right)\right) &= \theta^t q^*\left(\theta^t\right)
\end{aligned}
$$

A second-best solution:

1. When θ^t is not observable, then θ^H will not choose $q^*\left(\theta^H\right)$ because

$$
\begin{aligned}
\theta^H q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) &= \theta^H q\left(\theta^L\right) - \theta^L q\left(\theta^L\right) \\
&= \left(\theta^H - \theta^L\right) q\left(\theta^L\right) \\
&> 0 \\
&= \theta^H q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right)
\end{aligned}
$$

2. Firm decides $\left(p\left(q\left(\theta^L\right)\right), q\left(\theta^L\right), p\left(q\left(\theta^H\right)\right), q\left(\theta^H\right)\right)$ to maximize

$$
\max_{p\left(q\left(\theta^L\right)\right), q\left(\theta^L\right), p\left(q\left(\theta^H\right)\right), q\left(\theta^H\right)} (1-\alpha)\left[p\left(q\left(\theta^L\right)\right) - c\left(q\left(\theta^L\right)\right)\right] + \alpha\left[p\left(q\left(\theta^H\right)\right) - c\left(q\left(\theta^H\right)\right)\right]
$$

Two conditions have to be satisfied:

1. participation condition (PC)/Individual Rationality (IR)

$$
\begin{aligned}
(IR_1) &: \theta^L q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) \geq 0 \\
(IR_2) &: \theta^H q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right) \geq 0
\end{aligned}
$$

2. incentive compatibility (IC)

$$
\begin{aligned}
(IC_1) &: \theta^L q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) \geq \theta^L q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right) \\
(IC_2) &: \theta^H q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right) \geq \theta^H q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right)
\end{aligned}
$$

Preliminary results:

1. IR_1 is binding:

From IC$_2$, $\theta^H q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right) \geq \theta^H q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) \geq \theta^L q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right)$. Then if IR_1 is not binding, then IR$_2$ is not binding. Then firm can increase $p\left(q\left(\theta^L\right)\right)$ and $p\left(q\left(\theta^H\right)\right)$. Contradiction.

2. IC$_2$ is binding:

   Suppoe not. Then $\theta^H q\left(\theta^H\right) - p\left(q\left(\theta^H\right)\right) > \theta^H q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) \geq \theta^L q\left(\theta^L\right) - p\left(q\left(\theta^L\right)\right) = 0$. Then increases $p\left(q\left(\theta^H\right)\right)$ will not violate IC_1, IC_2 and IR_2. Contradiction.

3. $q\left(\theta^H\right) \geq q\left(\theta^L\right)$

   Summing up IC_1 and IC_2, we have $\theta^H(q\left(\theta^H\right) - q\left(\theta^L\right)) \geq \theta^L(q\left(\theta^H\right) - q\left(\theta^L\right))$. Since $\theta^H > \theta^L$, we have $q\left(\theta^H\right) \geq q\left(\theta^L\right)$.

4. IC$_1$ is redundant

   Since IC_2 is binding, we have $p\left(q\left(\theta^H\right)\right) - p\left(q\left(\theta^L\right)\right) = \theta^H(q\left(\theta^H\right) - q\left(\theta^L\right)) \geq \theta^L(q\left(\theta^H\right) - q\left(\theta^L\right))$ because $\theta^H > \theta^L$ and $q\left(\theta^H\right) \geq q\left(\theta^L\right)$.

5. IR$_2$ is redundant

   Since IC$_2$ is binding, $\theta^H q\left(\theta^H\right) - p(q\left(\theta^H\right)) = \theta^H q\left(\theta^L\right) - p(q\left(\theta^L\right)) \geq \theta^L q\left(\theta^L\right) - p(q\left(\theta^L\right)) = 0$.

Hence, the maximization problem now becomes

$$\max_{q\left(\theta^L\right), q\left(\theta^H\right)} \Pi = (1-\alpha)\left[\theta^L q\left(\theta^L\right) - c\left(q\left(\theta^L\right)\right)\right] + \alpha\left[\theta^H q\left(\theta^H\right) - \left(\theta^H - \theta^L\right) q\left(\theta^L\right) - c\left(q\left(\theta^H\right)\right)\right]$$

with FOC

$$\begin{aligned}
\frac{\partial \Pi}{\partial q\left(\theta^L\right)} &= 0 \Rightarrow \alpha\left[\theta^H - c'\left(q\left(\theta^H\right)\right)\right] = 0 \\
\frac{\partial \Pi}{\partial q\left(\theta^H\right)} &= 0 \Rightarrow (1-\alpha)\left[\theta^L - c'\left(q\left(\theta^L\right)\right)\right] - \alpha\left(\theta^H - \theta^L\right) = 0
\end{aligned}$$

so that

$$\begin{aligned}
\theta^H &= c'\left(q\left(\theta^H\right)\right) \\
\theta^L &= c'\left(q\left(\theta^L\right)\right) + \frac{\alpha}{1-\alpha}\left(\theta^H - \theta^L\right)
\end{aligned}$$

See appendix for continuous case.

Contiuous Version Assume quality $\theta \in \left[\underline{\theta}, \bar{\theta}\right]$ and quality $q(\theta)$ is increasing strictly.

$$U(\theta) = \theta q(\theta) - p(q(\theta)) \geq \theta q(\theta - \Delta) - p(q(\theta - \Delta))$$

and when $\Delta \to 0$, we have

$$\theta q'(\theta) - p'(q(\theta)) q'(\theta) = 0$$

Mirrlee's trick (1971):

$$\begin{aligned} U'(\theta) &= q(\theta) + q'(\theta) - p'(q(\theta)) q'(\theta) \\ &= q(\theta) \end{aligned}$$

so that

$$\begin{aligned} U(\theta) &= \textstyle\int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} + u(\underline{\theta}) \\ &= \textstyle\int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} \end{aligned}$$

where $u(\underline{\theta}) = 0$.

Profit from type θ consumer would be

$$p(q(\theta)) = -U(\theta) + \theta q(\theta)$$

and

$$\begin{aligned} \max_{\theta} \pi &= \textstyle\int_{\underline{\theta}}^{\bar{\theta}} \left[p(q(\theta)) - c(q(\theta))\right] f(\theta) d\theta \\ &= \textstyle\int_{\underline{\theta}}^{\bar{\theta}} \left[-U(\theta) + \theta q(\theta) - c(q(\theta))\right] f(\theta) d\theta \\ &= \textstyle\int_{\underline{\theta}}^{\bar{\theta}} \left[-\int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} + \theta q(\theta) - c(q(\theta))\right] f(\theta) d\theta \end{aligned}$$

Integration by parts

$$\frac{d \int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} \left(-1 + F(\theta)\right)}{d\theta} = \textstyle\int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} f(\theta) + q(\theta)\left[-1 + F(\theta)\right]$$

and having $F\left(\bar{\theta}\right) = 1$ and

$$\textstyle\int_{\underline{\theta}}^{\underline{\theta}} q\left(\tilde{\theta}\right) d\tilde{\theta} = 0$$

Hence, we have

$$\int_{\underline{\theta}}^{\bar{\theta}} \int_{\underline{\theta}}^{\theta} q\left(\tilde{\theta}\right) d\tilde{\theta} f\left(\theta\right) d\theta = -\int_{\underline{\theta}}^{\bar{\theta}} q\left(\tilde{\theta}\right) \left[-1 + F\left(\theta\right)\right] d\theta$$

and

$$\max_{q} \pi = \int_{\underline{\theta}}^{\bar{\theta}} \left[\theta q\left(\tilde{\theta}\right) - c\left(q\left(\theta\right)\right)\right] f\left(\theta\right) - q\left(\theta\right)\left[-1 + F\left(\theta\right)\right] d\theta$$

with FOC being

$$\begin{aligned} &\left[\theta - c'\left(q\left(\theta\right)\right)\right] f\left(\theta\right) - \left[1 - F\left(\theta\right)\right] = 0 \\ &\Rightarrow c'\left[q\left(\theta\right)\right] = \theta - \frac{1 - F\left(\theta\right)}{f\left(\theta\right)} \end{aligned}$$

such that

$$c'\left[q\left(\bar{\theta}\right)\right] = \bar{\theta}$$

A.4 Dynamic Games of Incomplete Information

A.4.1 Intuitive Criterion

Equilibrium domination:
m_j is equilibrium dominated for t_i if

$$U_s^*\left(t_i\right) > \max_{a_k} U_s\left(m_j, a_k, t_i\right)$$

Requirement 4: Intuitive criterion
If m_j is equilibrium dominated for t_i, then, if possible,

$$\mu\left(t_i \mid m_j\right) = 0$$

A.4.2 Applicaton: Limit pricing

Milgrom and Roberts (1982; Econometrica)
Two types of monopolist: High cost and Low cost $t \in \{H, L\}$
Period 1: monopolist sets price p.
Period 2: Observing the price, entrant makes decision to enter $e \in \{0, 1\}$. If entered, it is duopoly. Otherwise, it is monopoly.
Incumbent's strategy: $P : \{L, H\} \to \mathbb{R}_+$

Entrant's strategy: $E : \mathbb{R}_+ \to \{0, 1\}$
Entrant's belief $b_t : \mathbb{R}_+ \to [0, 1]$ or $b_L(p) + b_H(p) = 1$ for all p.
Incubment's payoff:

$$V(p, e, t) = \pi(p, t) + e\pi^d(t) + (1 - e)\pi^m(t)$$

Entrant's payoff:

$$U(p, e, t) = \begin{cases} \pi^e(t) & \text{if enter} \\ 0 & \text{otherwise} \end{cases}$$

Assumptions:

1. π is strictly concave in p.
2. Entry is not profitable for low cost monopolist: $\pi^e(H) > 0 > \pi^e(L)$
3. No entry is better for incumbent: $\pi^m(t) > \pi^d(t)$
4. Low cost is good for incumbent: $\pi^m(L) > \pi^m(H)$ and $\pi^d(L) > \pi^d(H)$
5. Low cost firm has better advantage in monopoly then duopoly

$$\pi^m(L) - \pi^d(L) \geq \pi^m(H) - \pi^d(H)$$

6. In the first stage, advantage of lower cost is higher when price is lower:

$$\pi(p, L) - \pi(p, H) > \pi(p', L) - \pi(p', H), \forall p < p'$$

Single crossing property (SCP):
Now combining 5 and 6, we have for all $p < p'$

$$\pi(p, L) - \pi(p', L) > \pi(p, H) - \pi(p', H)$$
$$\Rightarrow [\pi^m(L) + \pi(p, L)] - \left[\pi^d(L) + \pi(p', L)\right] > [\pi^m(H) + \pi(p, H)] - \left[\pi^d(H) + \pi(p', H)\right]$$
$$\Rightarrow V(p, 0, L) - V(p', 1, L) > V(p, 0, H) - V(p', 1, H)$$

Moreover, from 6, we have

$$\pi\left(p,L\right)-\pi\left(p',L\right)+\pi^{d}\left(L\right)-\pi^{d}\left(L\right)>\pi\left(p,H\right)-\pi\left(p',H\right)+\pi^{d}\left(H\right)-\pi^{d}\left(H\right)$$
$$\pi\left(p,L\right)-\pi\left(p',L\right)+\pi^{m}\left(L\right)-\pi^{m}\left(L\right)>\pi\left(p,H\right)-\pi\left(p',H\right)+\pi^{m}\left(H\right)-\pi^{m}\left(H\right)$$

so we have, for $e\in\{0,1\}$

$$V\left(p,e,L\right)-V\left(p',e,L\right)>V\left(p,e,H\right)-V\left(p',e,H\right)$$

SCP implies: if a low-cost incumbent suffers more from same level of entry and same price change and hence willing to accept deeper price cut to deter entry.
Perfect Bayesian Equilibrium requires:
(E1): $p\left(t\right)\in\arg\max_{P}V\left(p,E\left(p\right),t\right)$, $\forall t\in\{L,H\}$
(E2): $E\left(p\right)\in\arg\max_{e}E_{t}U\left(p,e^{t}\right)$, $t\in b\left(p\right)$
(E3): Bayes consistency:

$$\text{Pooling equilibrium: } P\left(L\right)=P\left(H\right)\Rightarrow b_{L}\left(P\left(L\right)\right)=b_{L}^{0}$$
$$\text{Correct Belief: } P\left(L\right)\neq P\left(H\right)\Rightarrow b_{L}\left(P\left(L\right)\right)=b_{H}\left(P\left(H\right)\right)=1$$

(E4): Intutive criterion:

$$b_{t}\left(p\right)=1$$

$$\text{if for } t\neq t'$$
$$V\left(p,0,t\right)\geq V\left(P\left(t\right),E\left(P\left(t\right)\right),t\right)$$
$$V\left(p,0,t'\right)<V\left(P\left(t'\right),E\left(P\left(t'\right),t'\right)\right)$$

First line: improve the type t; Second line: worse for type t'. Then it should be type t. There is a pooling equilibrium would be limiting price that incumbent charge at the price that deters entry.
Proof.
Let $p_{t}^{m}=\arg\max_{p}\Pi(p,t)$. Note that $p_{L}^{m}<p_{H}^{m}$.
Define $\tilde{b}_{L}$ be the belief that entrant is indifferent about entry at some equilibrium:

$$\tilde{b}_{L}\pi^{e}(L)+(1-\tilde{b}_{L})\pi^{e}(H)=0$$

Define $\underline{p}$ and $\bar{p}$ be the interval such that for any $\hat{p} \in [\underline{p}, \bar{p}]$

$$V(\hat{p}, 0, H) > V(p_H^m, 1, H)$$

Consider $p_L^m \geq \underline{p}$ and $b_L \geq \tilde{b}_L$.
Consider

$$P(L) = P(H) = p' \in [\underline{p}, p_L^m]$$

and

$$E(p) = \begin{cases} 0 & \text{for } p \leq p' \\ 1 & \text{for } p > p' \end{cases}$$

and

$$b_L(p) = \begin{cases} b_L^0 & \text{for } p \leq p' \\ 0 & \text{for } p > p' \end{cases}$$

These satisfy E1-E3. check E4.
For all $p < p'$, profits are lower for both type, so E4 doesn't apply.
Let p'' such that $V(p', 0, H) = V(p'', 0, H)$. (Concavity of V)
For all $p \in (p', p'']$, $V(p, 0, H) \geq V(p', 0, H)$. Hence, $b_L(p) = 0$.
For $p > p''$, by SCP, we hve $V(p'', 0, L) < V(p', 0, L)$ and hence by concavity of Π, we have $p'' > p_L^m$.
Hence, $V(p, 0, L) < V(p'', 0, L)$ and $V(p, 0, L) < V(p', 0, L)$ so that $b_L(p) = 0$.