EC 4101: Microeconomic Analysis III

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All relevant details on EC4101 are in the syllabus

3 suggestions

- Please follow the textbook and lectures closely.
- Use the material on the slides to guide you through the material in the textbook.
- Try solving all the assignments even if you might not be the designated student.

What is the job of an economic theorist?

- *The Raft of Medusa* (1819) - Jean Louis Théodore Géricault
- Moment from the aftermath of the wreck of the French naval frigate *Méduse*, which ran aground off the coast of today's Mauritania on July 5, 1816
- Take a real situation or story or thought … that exists and represent it in a useful manner

- Mars Rover Spirit (2004)
- The objective is to do something that is practically useful

What is EC 4101 about?

- Microeconomic theory is about modeling individual consumer and firm behavior in a mathematically to allow technically mature analysis.

- EC 3101 gives some description of these models – focuses on covering many topics rather than details
  - This description not enough for serious analysis

- EC4101 focuses on giving you the full technical description of those models
  - This description is useful for serious analysis

- Of course, that means we cannot cover as many topics.

Chapter 2

Mathematics for Microeconomics

- This file is a summary of the basic maths that you should know (but we do not have the time to go over in lecture). So keep it handy for future consultation.

- You will notice at a few places parenthesis such as "(" and ")" not aligned properly. This is not my fault – it's the slightly distorted pdf conversion.

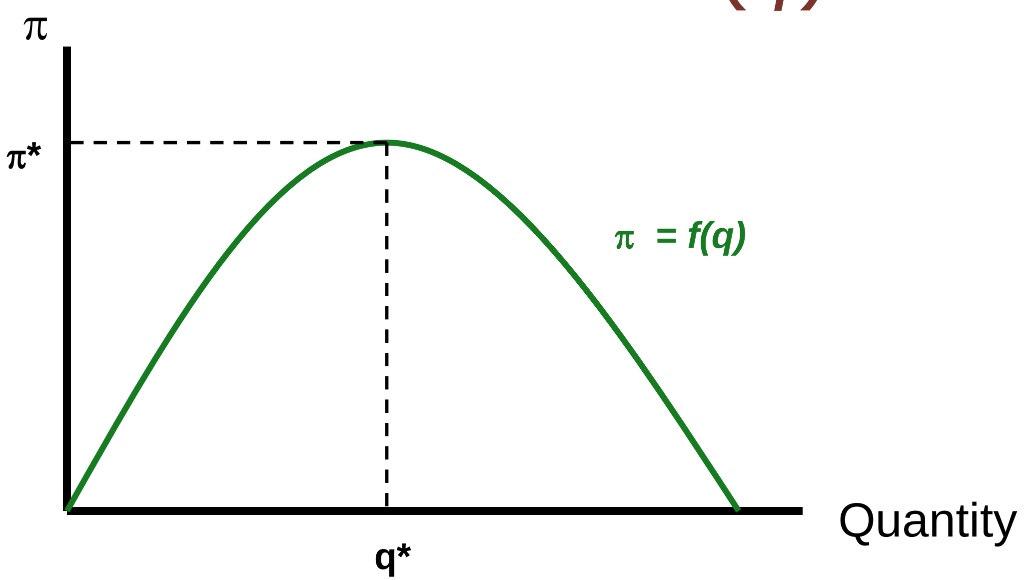
The Mathematics of Optimization

- Economic theories assume that an economic agent is seeking to find the optimal value of some function
  - consumers seek to maximize utility
  - firms seek to maximize profit
- This chapter reviews the mathematics that go into these problems

Functions with One Variable

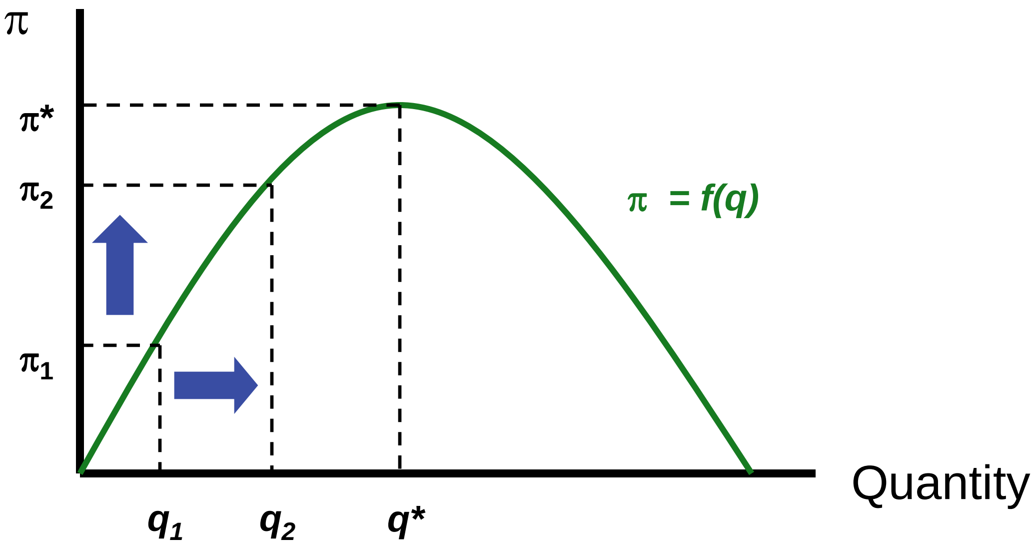
- Simple example: Manager of a firm wants to maximize profits

$$\pi = f(q)$$



Functions with One Variable

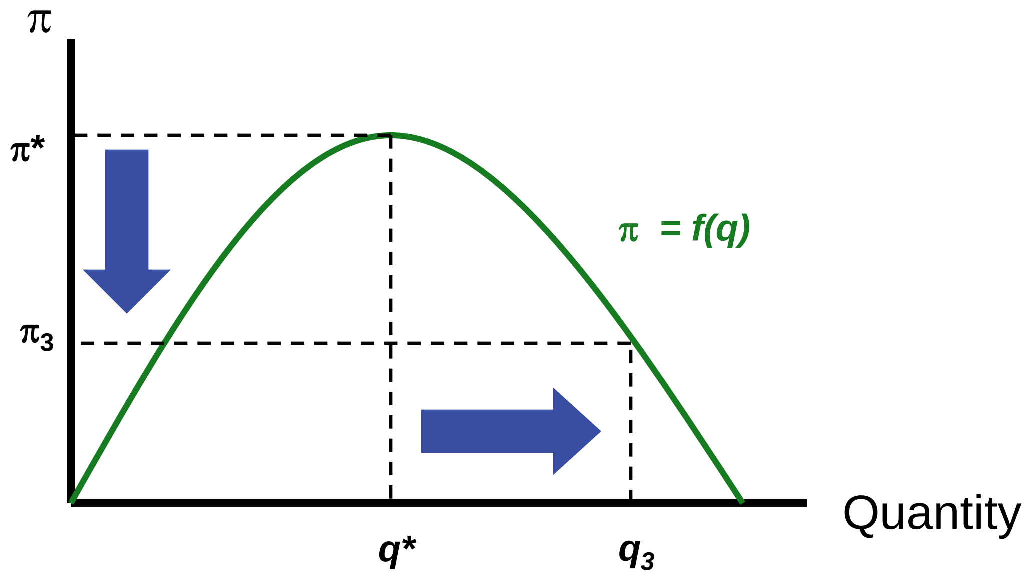
- Vary *q* to see where maximum profit occurs
  - an increase from q_1 to q_2 leads to a rise in $\boldsymbol{\pi}$



$$\frac{\Delta\pi}{\Delta q} > 0$$

Functions with One Variable

- If output is increased beyond q^*, profit will decline
  - an increase from q^* to q_3 leads to a drop in π



$$\frac{\Delta\pi}{\Delta q} < 0$$

Derivatives

- The derivative of $\pi = f(q)$ is the limit of $\Delta\pi/\Delta q$ for very small changes in q

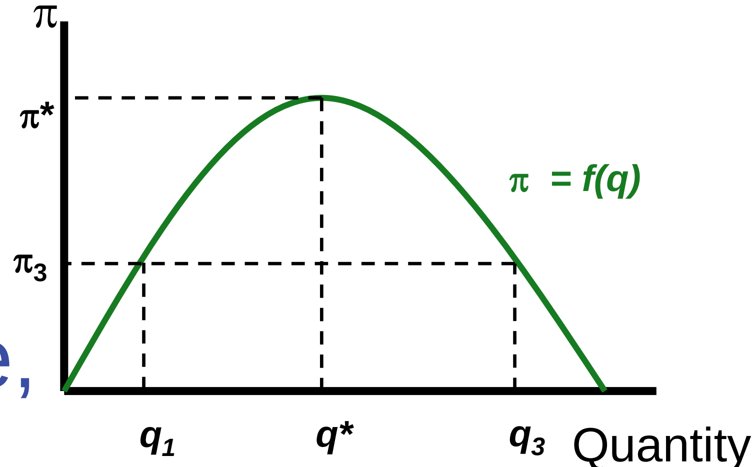
$$\frac{d\pi}{dq} = \frac{df}{dq} = \lim_{h \to 0} \frac{f(q_1 + h) - f(q_1)}{h}$$

- The value depends on the value of q_1

Value of a Derivative at a Point

- The evaluation of the derivative at the point $q = q_1$ can be denoted

$$\left.\frac{d\pi}{dq}\right|_{q=q_1}$$



- In our previous example,

$$\boxed{\left.\frac{d\pi}{dq}\right|_{q=q_1} > 0} \qquad \boxed{\left.\frac{d\pi}{dq}\right|_{q=q_3} < 0} \qquad \boxed{\left.\frac{d\pi}{dq}\right|_{q=q^*} = 0}$$

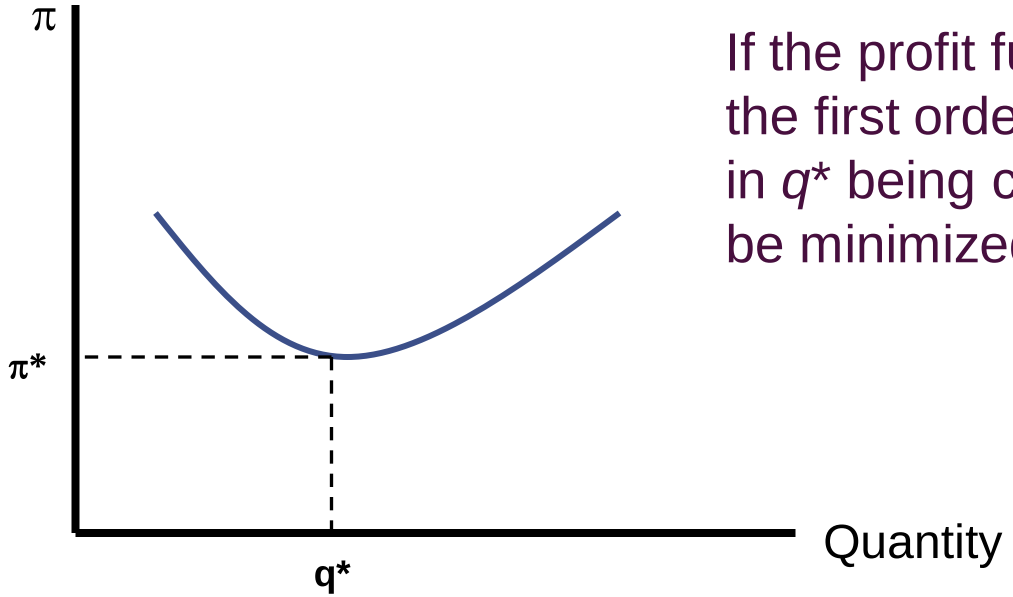
First Order Condition for a Maximum

- For a function of one variable to attain its maximum value at some point, the derivative at that point must be zero

$$\left.\frac{df}{dq}\right|_{q=q*} = 0$$

Second Order Conditions

- The first order condition ($d\pi/dq$) is a <u>necessary</u> condition for a maximum, but it is not a <u>sufficient</u> condition



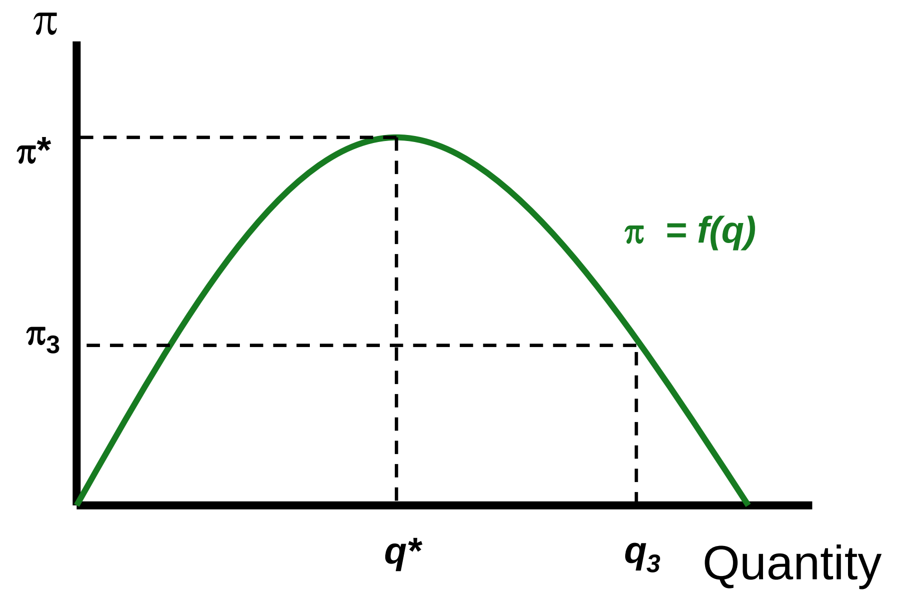
If the profit function was u-shaped, the first order condition would result in q^* being chosen and π would be minimized

Second Order Conditions

- This must mean that, in order for *q** to be the optimum,

$$\frac{d\pi}{dq} > 0 \text{ for } q < q^* \quad \text{and} \quad \frac{d\pi}{dq} < 0 \text{ for } q > q^*$$

- At *q**, $d\pi/dq$ must be decreasing
  - the derivative of $d\pi/dq$ must be negative at *q**



Second Derivatives

- The derivative of a derivative is called a <u>second derivative</u>
- The second derivative can be denoted by

$$\frac{d^2\pi}{dq^2} \text{ or } \frac{d^2 f}{dq^2} \text{ or } f''(q)$$

Second Order Condition

- The second order condition to represent a (local) maximum is

$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = f''(q)\Big|_{q=q^*} < 0$$

Rules for Finding Derivatives

1. If b is a constant, then $\dfrac{db}{dx} = 0$

2. If b is a constant, then $\dfrac{d[bf(x)]}{dx} = bf'(x)$

3. If b is constant, then $\dfrac{dx^b}{dx} = bx^{b-1}$

4. $\dfrac{d \ln x}{dx} = \dfrac{1}{x}$

Rules for Finding Derivatives

5. $\dfrac{da^x}{dx} = a^x \ln a$ for any constant a

– a special case of this rule is $de^x/dx = e^x$

Rules for Finding Derivatives

- Suppose that $f(x)$ and $g(x)$ are two functions of x and $f'(x)$ and $g'(x)$ exist
- Then

$$6.\ \frac{d[f(x)+g(x)]}{dx}=f'(x)+g'(x)$$

$$7.\ \frac{d[f(x)\cdot g(x)]}{dx}=f(x)g'(x)+f'(x)g(x)$$

Rules for Finding Derivatives

8. $$\frac{d\left(\dfrac{f(x)}{g(x)}\right)}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

provided that $g(x) \neq 0$

Rules for Finding Derivatives

- If $y = f(x)$ and $x = g(z)$ [so $y = f(g(z))$] and if both $f'(x)$ and $g'(x)$ exist, then:

9. $$\frac{dy}{dz} = \frac{dy}{dx} \cdot \frac{dx}{dz} = \frac{df}{dx} \cdot \frac{dg}{dz}$$

  - this is called the <u>chain rule</u>
  - allows us to study how one variable (z) affects another variable (y) through its influence on some intermediate variable (x)

Rules for Finding Derivatives

- Some examples of the chain rule include

10. $$\frac{de^{ax}}{dx} = \frac{de^{ax}}{d(ax)} \cdot \frac{d(ax)}{dx} = e^{ax} \cdot a = ae^{ax}$$

11. $$\frac{d[ln(ax)]}{dx} = \frac{d[ln(ax)]}{d(ax)} \cdot \frac{d(ax)}{dx} = \frac{1}{ax} \cdot a = \frac{1}{x}$$

12. $$\frac{d[\ln(x^2)]}{dx} = \frac{d[\ln(x^2)]}{d(x^2)} \cdot \frac{d(x^2)}{dx} = \frac{1}{x^2} \cdot 2x = \frac{2}{x}$$

Example of Profit Maximization

- Suppose that the relationship between profit and output is

$$\pi = 1{,}000q - 5q^2$$

- The first order condition for a maximum is

$$d\pi/dq = 1{,}000 - 10q = 0$$

$$q^* = 100$$

- Since the second derivative is always -10, $q = 100$ is a <u>global maximum</u>

Functions of Several Variables

- Most goals of economic agents depend on several variables
  - trade-offs must be made
- The dependence of one variable (y) on a series of other variables ($x_1, x_2, \ldots, x_n$) is denoted by

$$y = f(x_1, x_2, \ldots, x_n)$$

Partial Derivatives

- The partial derivative of y with respect to x_1 is denoted by

$$\frac{\partial y}{\partial x_1} \text{ or } \frac{\partial f}{\partial x_1} \text{ or } f_{x_1} \text{ or } f_1$$

  – in calculating the partial derivative, all of the other x's are held constant

Partial Derivatives

- A more formal definition of the partial derivative is

$$\left.\frac{\partial f}{\partial x_1}\right|_{\overline{x}_2,\ldots,\overline{x}_n} = \lim_{h \to 0} \frac{f(x_1 + h, \overline{x}_2, \ldots, \overline{x}_n) - f(x_1, \overline{x}_2, \ldots, \overline{x}_n)}{h}$$

Calculating Partial Derivatives

1. If $y = f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = 2ax_1 + bx_2 \quad \text{and}$$

$$\frac{\partial f}{\partial x_2} = f_2 = bx_1 + 2cx_2$$

2. If $y = f(x_1, x_2) = e^{ax_1 + bx_2}$, then

$$\frac{\partial f}{\partial x_1} = f_1 = ae^{ax_1 + bx_2} \quad \text{and} \quad \frac{\partial f}{\partial x_2} = f_2 = be^{ax_1 + bx_2}$$

Calculating Partial Derivatives

3. If $y = f(x_1, x_2) = a\ln x_1 + b\ln x_2$, then

$$\frac{\partial f}{\partial x_1} = f_1 = \frac{a}{x_1} \text{ and } \frac{\partial f}{\partial x_2} = f_2 = \frac{b}{x_2}$$

Partial Derivatives

- Partial derivatives are the mathematical expression of the *ceteris paribus* assumption
  - show how changes in one variable affect some outcome when other influences are held constant

Second-Order Partial Derivatives

- The partial derivative of a partial derivative is called a <u>second-order partial derivative</u>

$$\frac{\partial(\partial f / \partial x_i)}{\partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{ij}$$

Young’s Theorem

- Under general conditions, the order in which partial differentiation is conducted to evaluate second-order partial derivatives does not matter

$$f_{ij} = f_{ji}$$

Functions of Several Variables

- Suppose an agent wishes to maximize

$$y = f(x_1, x_2, \ldots, x_n)$$

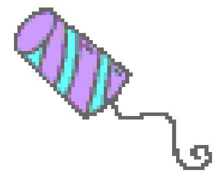
- The change in y from a change in x_1 (holding all other x's constant) is

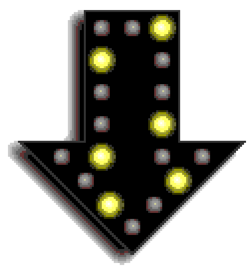
$$dy = \frac{\partial f}{\partial x_1} dx_1 = f_1 dx_1$$

  – the change in y is equal to the change in x_1 times the slope (measured in the x_1 direction)

Party time! Let’s make the change bigger

$$dy = \frac{\partial f}{\partial x_1} dx_1 = f_1 dx_1$$





$$\Delta y = \frac{\partial f}{\partial x_1} \Delta x_1 = f_1 \Delta x_1$$

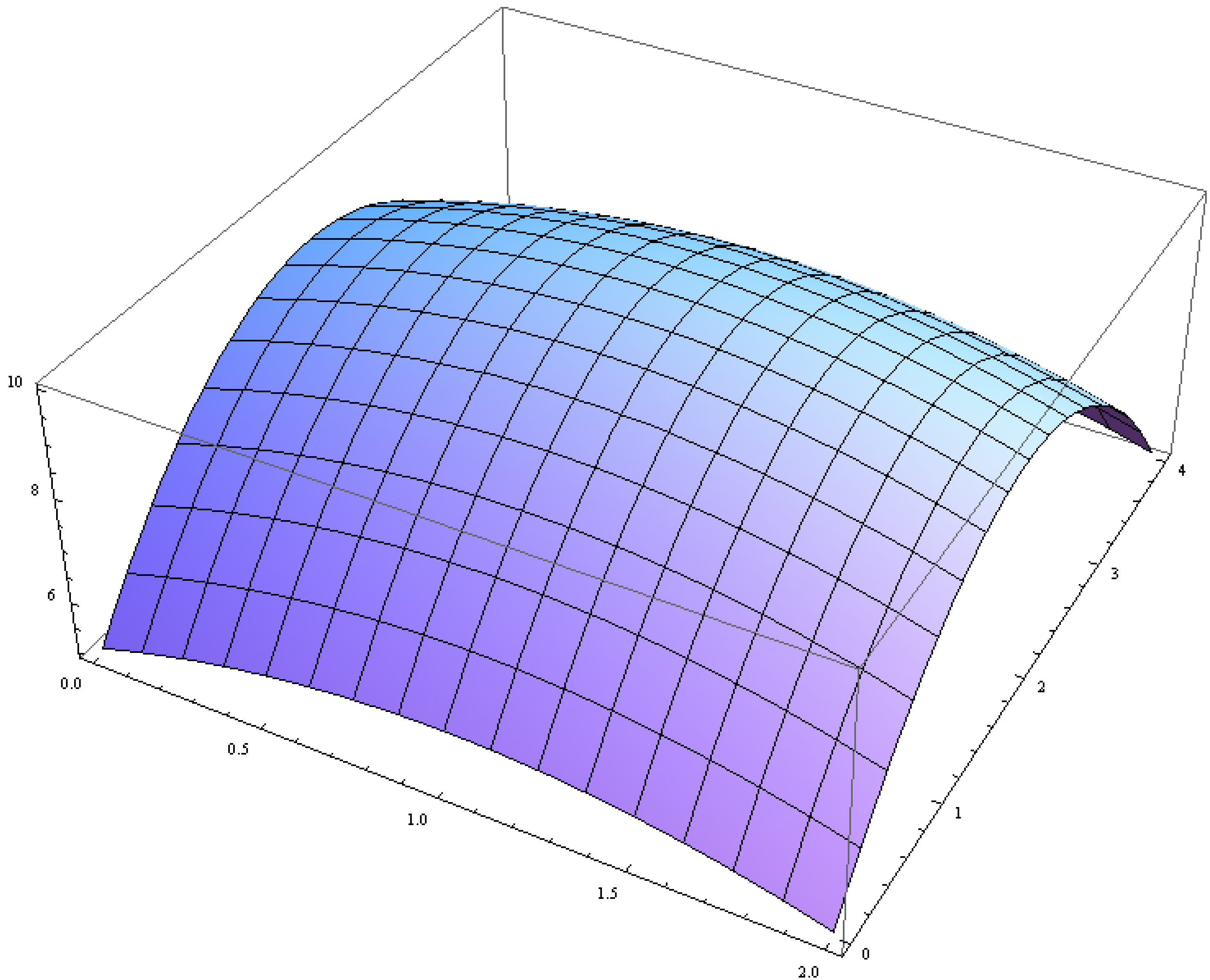
Total Differential

- Suppose that $y = f(x_1, x_2, \ldots, x_n)$
- If all x's are varied by a small amount, the total effect on y will be

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \ldots + \frac{\partial f}{\partial x_n} dx_n$$

$$dy = f_1 dx_1 + f_2 dx_2 + \ldots + f_n dx_n$$

$$\Delta y = f_1 \Delta x_1 + f_2 \Delta x_2 + \ldots + f_n \Delta x_n$$



10
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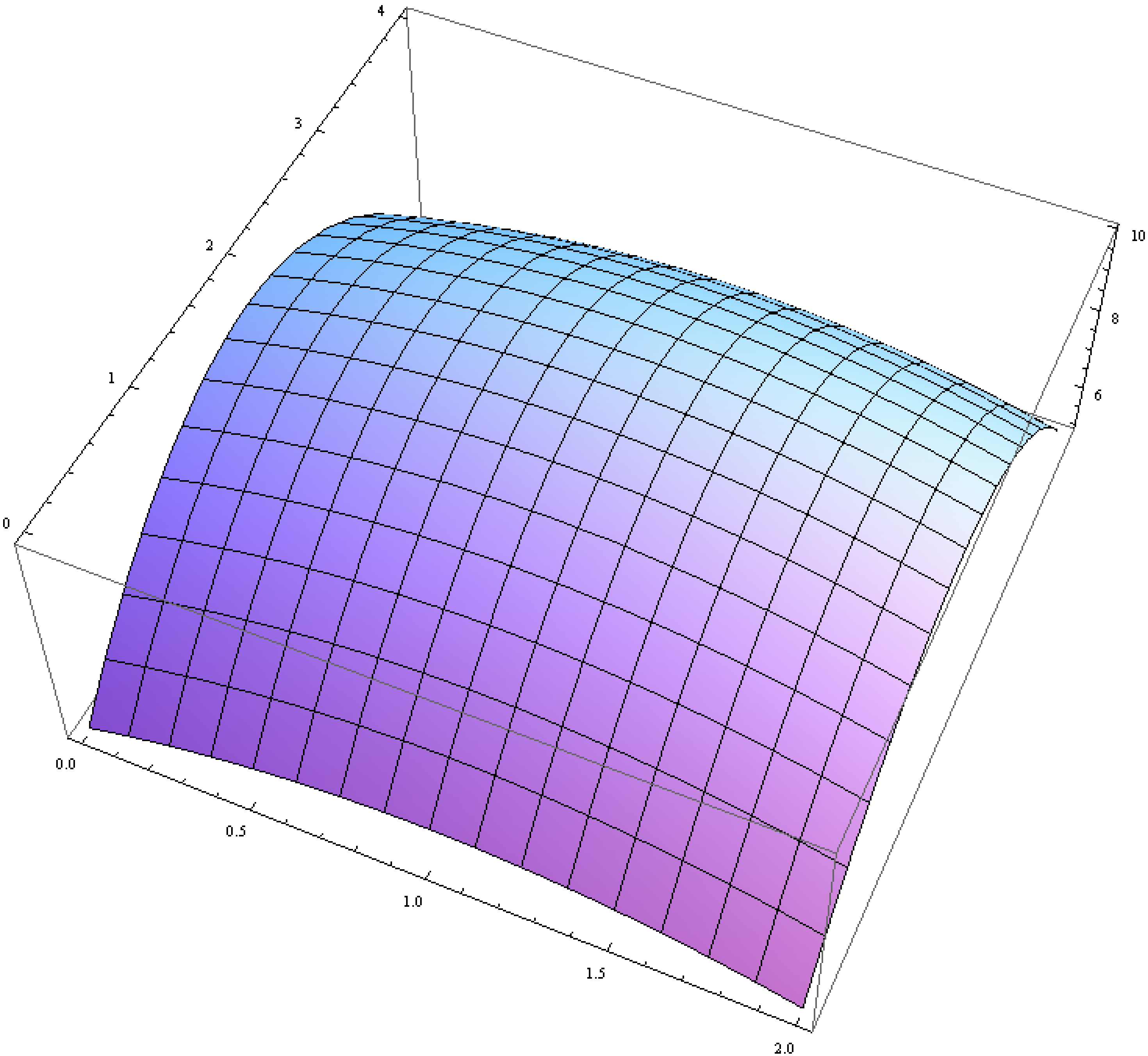

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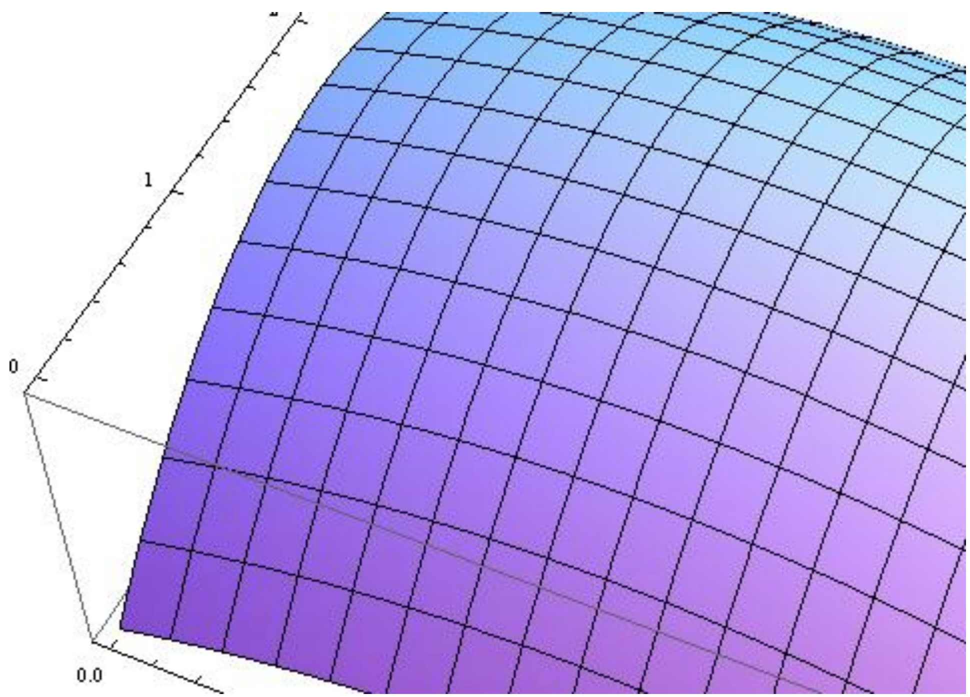
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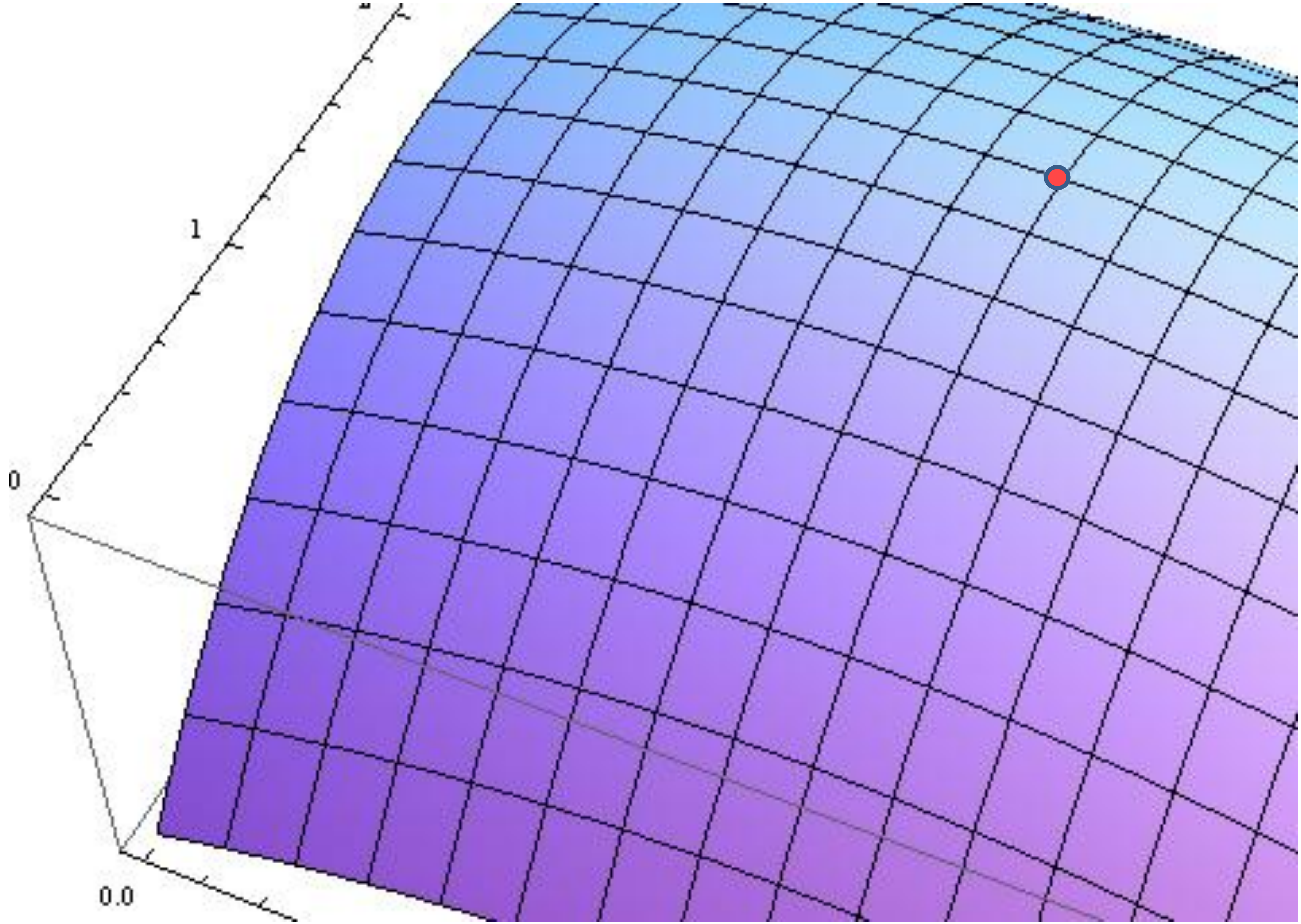
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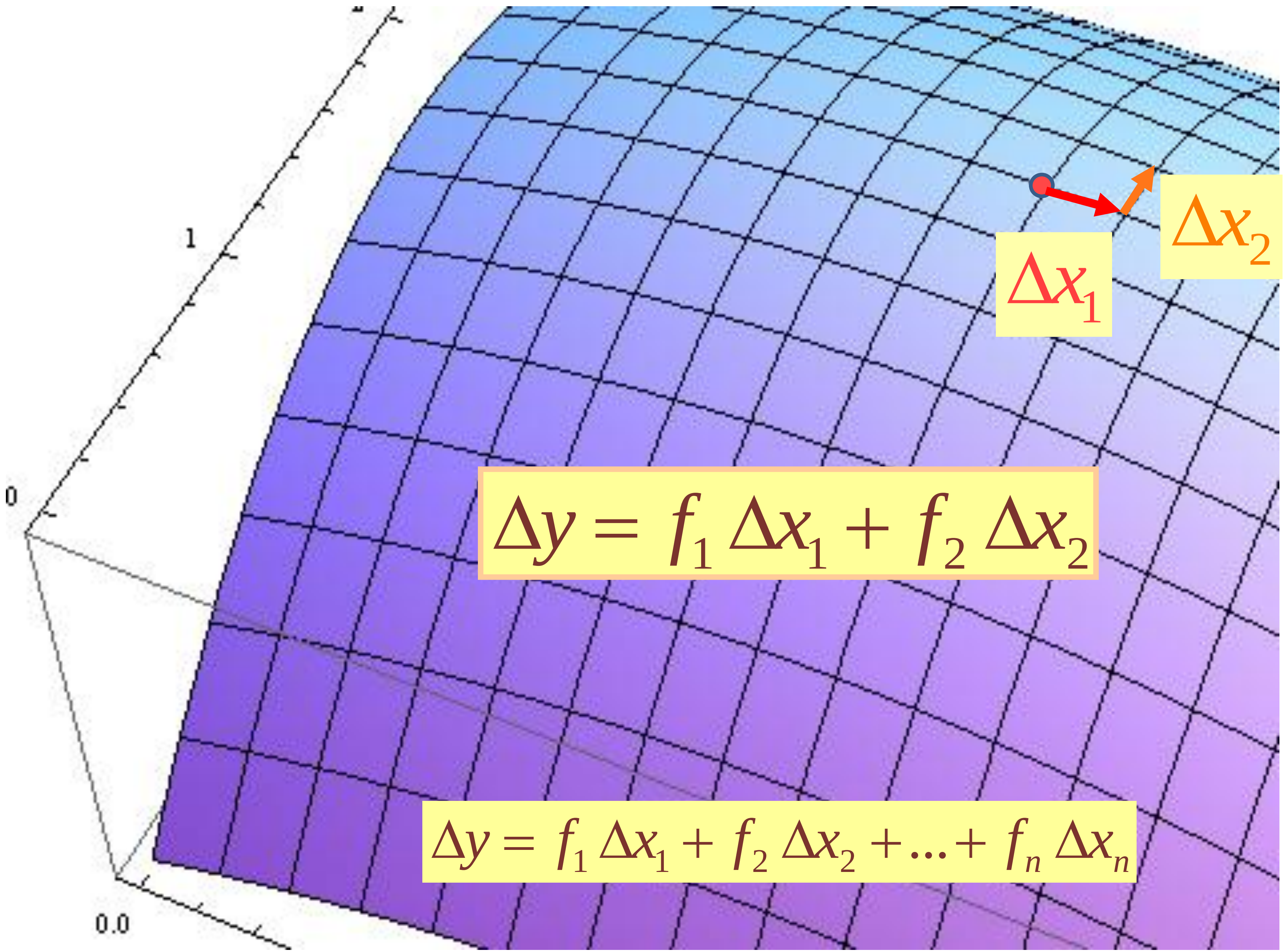


$$\Delta y = f_1\,\Delta x_1 + f_2\,\Delta x_2$$

$$\Delta y = f_1\,\Delta x_1 + f_2\,\Delta x_2 + \ldots + f_n\,\Delta x_n$$

$$\boxed{\Delta y = f_1 \Delta x_1 + f_2 \Delta x_2 + \ldots + f_n \Delta x_n}$$

First-Order Condition for a Maximum

- A necessary condition for a maximum of the function $f(x_1,x_2,\ldots,x_n)$ is that $dy = 0$ for any combination of small changes in the x's
  - this can only be true if

$$f_1 = f_2 = \ldots = f_n = 0$$

- A point where this condition holds is called a critical point

Second-Order Conditions

- This condition is not <u>sufficient</u> to ensure a maximum
  - we need to examine the second-order partial derivatives of the function *f*
  - conditions that will make *f* concave would be sufficient for a maximum

Finding a Maximum

- Suppose that y is a function of x_1 and x_2

$$y = -(x_1 - 1)^2 - (x_2 - 2)^2 + 10$$

$$y = -x_1^2 + 2x_1 - x_2^2 + 4x_2 + 5$$

- First-order conditions imply that

$$\frac{\partial y}{\partial x_1} = -2x_1 + 2 = 0$$

$$\frac{\partial y}{\partial x_2} = -2x_2 + 4 = 0$$

OR

$$x_1^* = 1$$

$$x_2^* = 2$$

Implicit Functions

- An “explicit” function which is shown with a dependent variable (y) as a function of one or more independent variables (x) such as

$$y = mx + b$$

can be written as an “implicit” function

$$y - mx - b = 0$$

$$f(x,y,m,b) = 0$$

Derivatives from Implicit Functions

- It will sometimes be helpful to compute derivatives directly from implicit functions without solving for one of the variables directly
  - the total differential of $g(x,y) = 0$ is

$$0 = g_x dx + g_y dy$$

  - this means that

$$\frac{dy}{dx} = -\frac{g_x}{g_y}$$

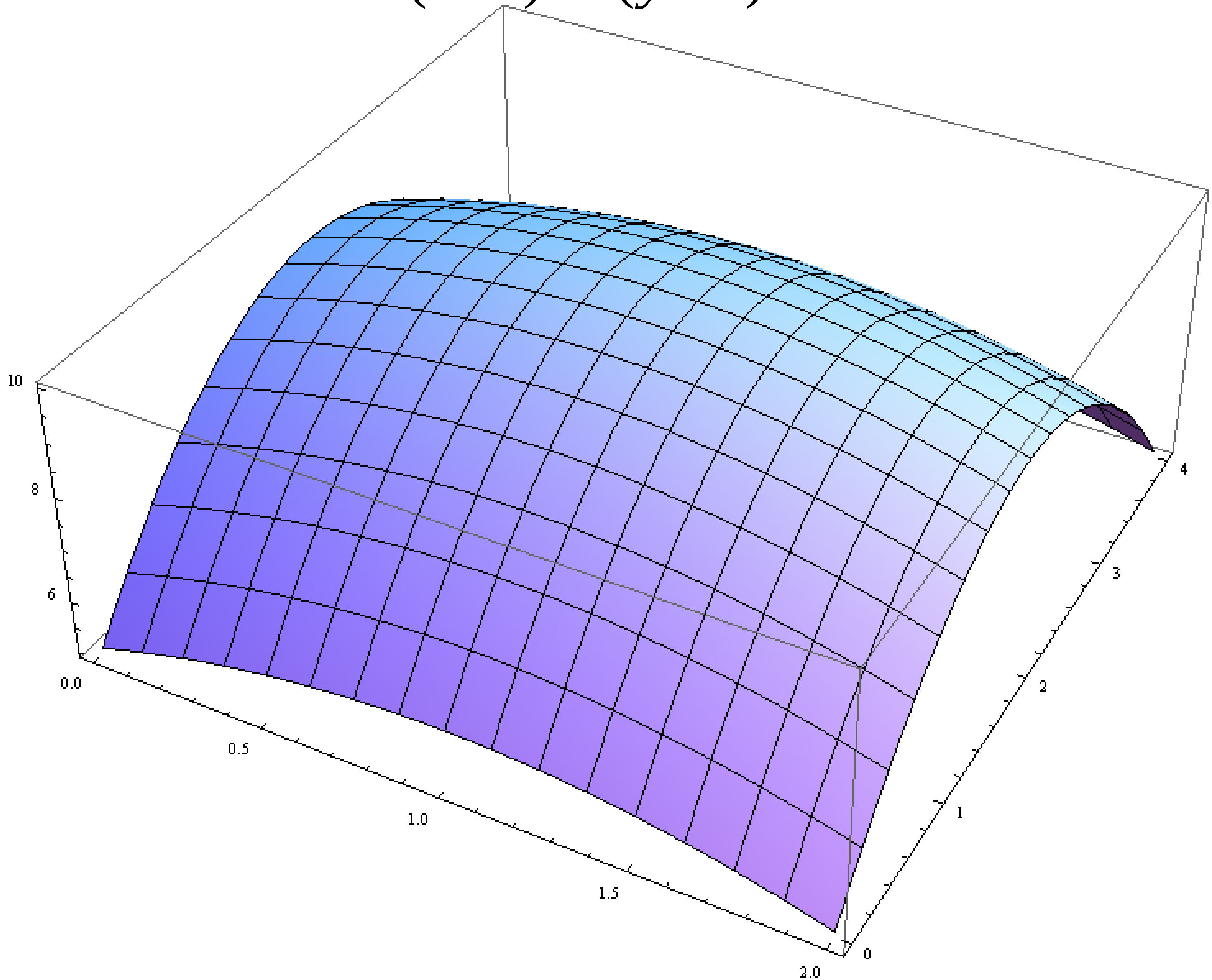
Implicit Function Theorem

- It may not always be possible to locally solve implicit functions of the form $g(x,y)=0$ for unique explicit functions of the form $y = f(x)$

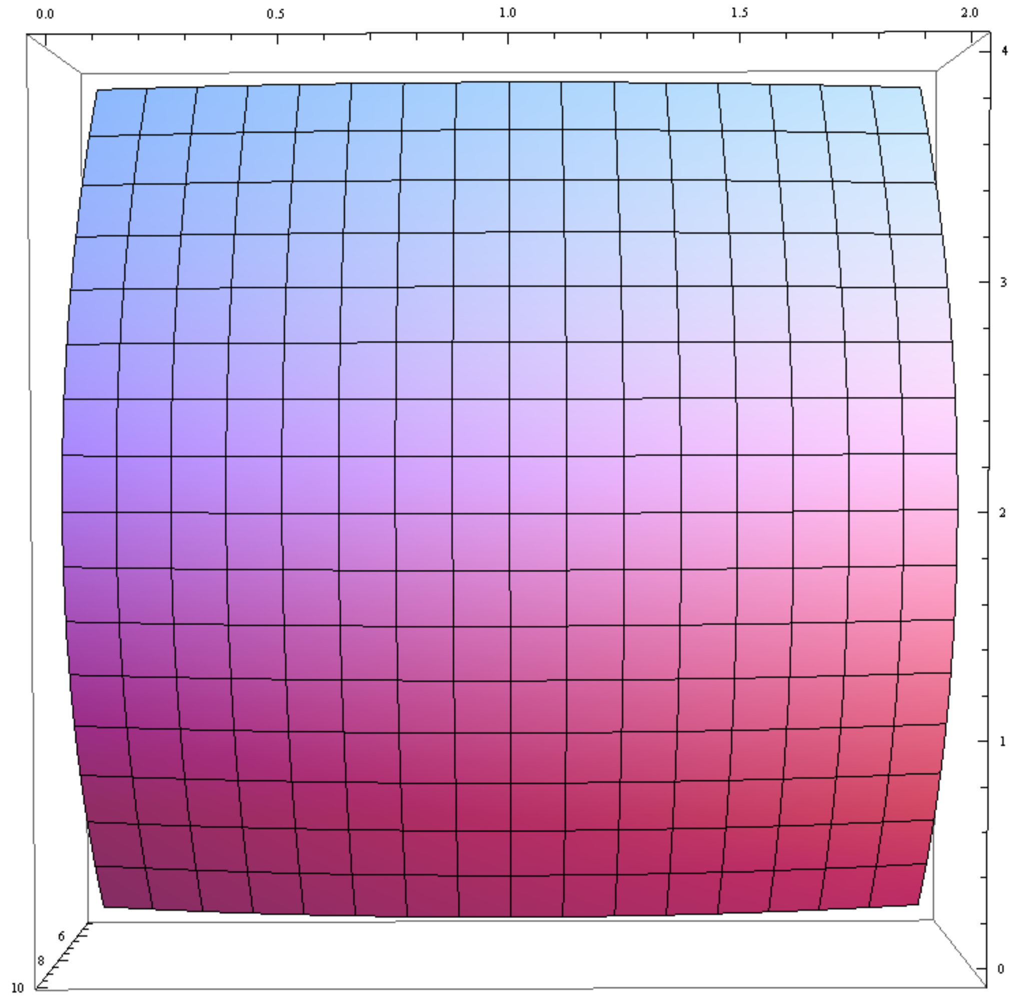
$$\text{Sufficient condition for } y = f(x) \text{ to exist :}$$

$$g_y \neq 0$$

$$10-(x-1)^2-(y-2)^2=0$$



$$10-(x-1)^2-(y-2)^2=0$$



Implicit Function Theorem

- It may not always be possible to locally solve implicit functions of the form $g(x,y)=0$ for unique explicit functions of the form $y = f(x)$

$$\text{Sufficient condition for } y = f(x) \text{ to exist :}$$

$$g_y \neq 0$$

The Envelope Theorem

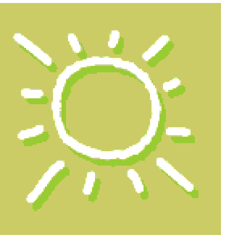
- The envelope theorem concerns how the optimal value for a function changes when a parameter of the function changes
  - this is easiest to see by using an example

The Envelope Theorem

- Suppose that *y* (ice cream seller's profit) is a function of *x* (ice cream output)

$$y = -x^2 + ax$$

- If *a* (temperature) is assigned a specific value, then *y* becomes a function of *x* only and the value of *x* that maximizes *y* can be calculated

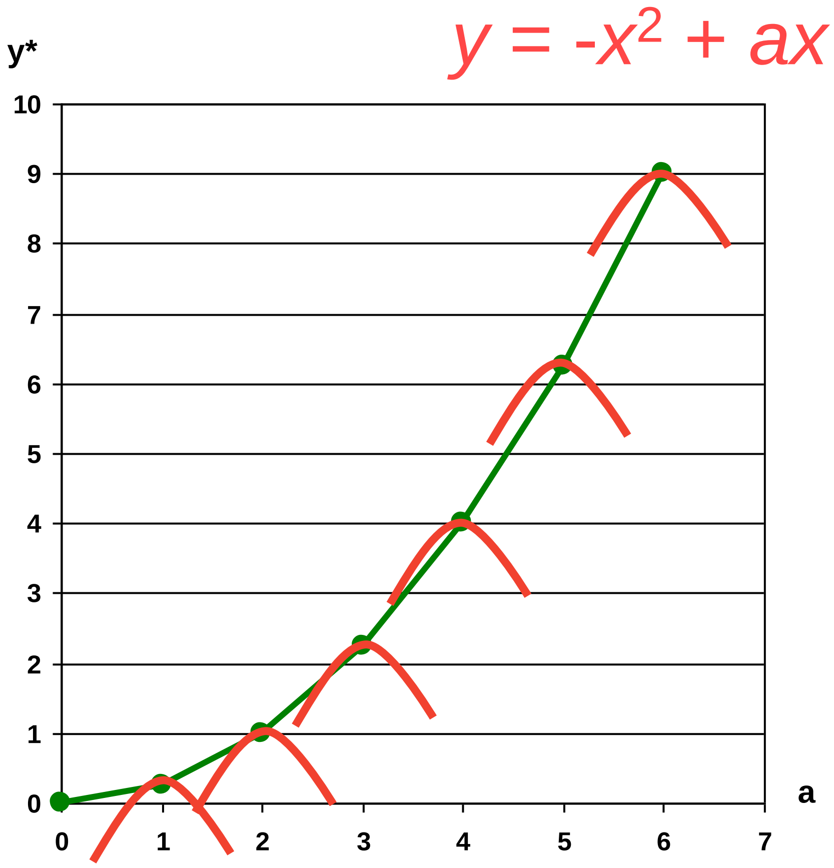


How does the profit depend on temperature? Use the Envelope Theorem

Optimal Values of x and y for Alternative Values of a

Value of a	Value of x^*	Value of y^*(Profit)
0	0	0
1	1/2	1/4
2	1	1
3	3/2	9/4
4	2	4
5	5/2	25/4
6	3	9

The Envelope Theorem



As *a* increases, the maximal value for *y* increases

The relationship between *a* and *y* is quadratic

$$\frac{dy^*}{da} = ?$$

The Envelope Theorem

- Suppose we are interested in how y^* changes as a changes

$$\frac{dy^*}{da} = ?$$

- There are two ways we can do this
  - calculate the slope of y^* directly
  - apply envelope theorem

The Direct Approach

- To calculate the slope of the function, we must solve for the optimal value of *x* for any value of *a*

$$dy/dx = -2x + a = 0$$

$$x^* = a/2$$

- Substituting, we get

$$y^* = -(x^*)^2 + a(x^*) = -(a/2)^2 + a(a/2)$$

$$y^* = -a^2/4 + a^2/2 = a^2/4$$

The Direct Approach

- Therefore,

$$dy^*/da = 2a/4 = a/2$$

- We can save time by using the envelope theorem (x* may not also be explicit)
  - for small changes in a, dy^*/da can be computed by holding x at x^* and calculating $\partial y/\partial a$ directly from y

The Envelope Theorem Way

$$\partial y / \partial a = x$$

- Holding $x = x^*$

$$\partial y / \partial a = x^* = a/2$$

- This is the same result found earlier

The Envelope Theorem

- The change in the optimal value of a function with respect to a parameter of that function can be found by partially differentiating the objective function while holding *x* (or several *x*'s) at its optimal value

$$\frac{dy^*}{da} = \left.\frac{\partial y}{\partial a}\right|_{x=x^*(a)}$$

The Math of Envelope Theorem

- How did the formula come about? We have from maximizing $y = f\,[x_1, x_2, a]$ with respect to x_1 and x_2

$$y^* = f\,(x_1{}^*(a),\, x_2{}^*(a), a)$$

- Taking derivative w.r.t. a

$$\frac{dy^*}{da} = \underbrace{\cancel{\frac{\partial f}{\partial x_1}}}_{0} \cdot \frac{dx_1}{da} + \underbrace{\cancel{\frac{\partial f}{\partial x_2}}}_{0} \cdot \frac{dx_2}{da} + \frac{\partial f}{\partial a}$$

The Envelope Theorem: Extention

- This can be extended to the case where y is a function of several variables

$$y = f(x_1,\ldots x_n,a)$$

- Finding an optimal value for *y* would consist of solving *n* first-order equations

$$\partial y/\partial x_i = 0 \quad (i = 1,\ldots,n)$$

The Envelope Theorem

- Optimal values for these x's would be a function of a

$$x_1^* = x_1^*(a)$$
$$x_2^* = x_2^*(a)$$
$$\vdots$$
$$x_n^* = x_n^*(a)$$

The Envelope Theorem

- Substituting into the original objective function gives us the optimal value of *y* (*y**)

$$y^* = f\,[x_1^*(a), x_2^*(a), \ldots, x_n^*(a), a]$$

- Differentiating yields

$$\frac{dy^*}{da} = \frac{\partial f}{\partial x_1} \cdot \frac{dx_1}{da} + \frac{\partial f}{\partial x_2} \cdot \frac{dx_2}{da} + \ldots + \frac{\partial f}{\partial x_n} \cdot \frac{dx_n}{da} + \frac{\partial f}{\partial a}$$

The Envelope Theorem

- Because of first-order conditions, all terms except $\partial f/\partial a$ are equal to zero if the x's are at their optimal values

- Therefore,

$$\frac{dy^*}{da} = \frac{\partial f}{\partial a}$$

Constrained Maximization

- Suppose that we wish to find the values of $x_1, x_2, \ldots, x_n$ that maximize

$$y = f(x_1, x_2, \ldots, x_n)$$

subject to a constraint

$$g(x_1, x_2, \ldots, x_n) = 0$$

Lagrangian Multiplier Method

- The Lagrangian multiplier method starts with setting up the expression

$$\mathscr{L} = f(x_1, x_2, \ldots, x_n) + \lambda g(x_1, x_2, \ldots, x_n)$$

  - λ is called a Lagrangian multiplier

$$\mathscr{L} = f(x_1, x_2, \ldots, x_n) + \lambda g(x_1, x_2, \ldots, x_n)$$

Lagrangian Multiplier Method

- First-Order Conditions

$$\partial\mathscr{L}/\partial x_1 = f_1 + \lambda g_1 = 0$$

$$\partial\mathscr{L}/\partial x_2 = f_2 + \lambda g_2 = 0$$

$$\vdots$$

$$\partial\mathscr{L}/\partial x_n = f_n + \lambda g_n = 0$$

$$\partial\mathscr{L}/\partial\lambda = g(x_1, x_2, \ldots, x_n) = 0$$

Interpretation of Lagrangian Multiplier

- Rate at which the maximum increases as the constraint is relaxed – shadow price for the constraint

We will come back to this in just a bit for a good understanding

Constrained Maximization

- Suppose a farmer had a certain length of fence (P) and wished to enclose the largest possible rectangular area
  - let x and y be the lengths of the sides
- Problem: choose x and y to maximize the area ($A = x \cdot y$) subject to the constraint that the perimeter is fixed at $P = 2x + 2y$

Constrained Maximization

- Setting up the Lagrangian multiplier:

$$\mathscr{L} = x \cdot y + \lambda(P - 2x - 2y)$$

- The first-order conditions for a maximum are

$$\partial\mathscr{L}/\partial x = y - 2\lambda = 0$$

$$\partial\mathscr{L}/\partial y = x - 2\lambda = 0$$

$$\partial\mathscr{L}/\partial\lambda = P - 2x - 2y = 0$$

Constrained Maximization

- Since $y/2 = x/2 = \lambda$, x must be equal to y
  - the field should be square

- Since $x = y$ and $y = 2\lambda$, we can use the constraint to show that

$$x = y = P/4$$

$$\lambda = P/8$$

Constrained Maximization

- Interpretation of the Lagrangian multiplier
  - λ suggests that an extra yard of fencing would add $P/8$ to the area
  - The Lagrangian multiplier provides information about the implicit value of the constraint

Constrained Maximization & Envelope Theorem

- Suppose that we want to maximize

$$y = f(x_1,\ldots,x_n;a)$$

subject to the constraint

$$g(x_1,\ldots,x_n;a) = 0$$

- Solve by setting partial derivatives of Lagrangian equal to 0

Constrained Maximization & Envelope Theorem

- It can be shown that

$$dy^*/da = \partial\mathscr{L}/\partial a \text{ at } (x_1^*,\ldots,x_n^*;a)$$

  – the change in the maximal value of *y* from a change in *a* can be found by partially differentiating $\mathscr{L}$ and evaluating the partial derivative at the optimal point

Inequality Constraints (Not in the textbook)

- In some economic problems the constraints need not hold exactly
- Suppose we seek to maximize $y = f(x_1,x_2)$ subject to

$$g_1\,(x_1,x_2) \geq 0,$$

$$g_2\,(x_1,x_2) \geq 0$$

Lagrangian

- We define
- $\mathscr{L} = f(x_1, x_2) + \lambda_1\, g_1\,(x_1,x_2) + \lambda_2\, g_2\,(x_1,x_2)$
- Now we write down the first order conditions for this Lagrangian
- In this case the multipliers have specific signs: $\lambda_1, \lambda_2 \geq 0$
- Complementary slackness: at least one of $g_i\,(x_1,x_2)$ or λ_i must be zero at solution

Kuhn-Tucker Conditions

- Sometimes we have a standard [illegible]trained optimization problem wit[illegible]nal constraints that t[illegible]negative.
- In that [illegible] non-negativity cons[illegible]ality constraints and work with t[illegible]gragian the standard way

Skip

Second Order Conditions - Functions of One Variable

- Let $y = f(x)$
- A necessary condition for a maximum is that

$$dy/dx = f'(x) = 0$$

  - to ensure that the point is a maximum, y must be decreasing for movements away from it

Second Order Conditions – Sufficiency Condition

$$\frac{d^2y}{dx^2} = f''(x) < 0$$

- This means that the function *f* must have a concave shape at the critical point

Second Order Conditions - Functions of Two Variables

- Suppose that $y = f(x_1, x_2)$
- First order conditions for a maximum are

$$\partial y/\partial x_1 = f_1 = 0$$

$$\partial y/\partial x_2 = f_2 = 0$$

  – to ensure that the point is a maximum, y must diminish for movements in any direction away from the critical point

Second Order Conditions - Functions of Two Variables

$$f_{11} < 0, f_{11}f_{22} - f_{12}^2 > 0$$

The rest of the sufficiency conditions will be on a need-to-know basis ☺

Quasi-Concavity

- A function U is quasi-concave if for each number c the following set is convex

$$\left[(x,y)\,|\,U(x,y)\geq c\right]$$

Duality

- Any constrained maximization problem has a dual problem in constrained minimization
  - focuses attention on the constraints in the original problem

To be studied later

Integration

- Integration is the inverse of differentiation
  - let $F(x)$ be the integral of $f(x)$
  - then $f(x)$ is the derivative of $F(x)$

$$\frac{dF(x)}{dx} = F'(x) = f(x)$$

Integration

- We denote an integral as

$$F(x) = \int f(x)\,dx$$

- If $f(x) = x$ then

$$F(x) = \int f(x)\,dx = \int x\,dx = \frac{x^2}{2} + C$$

  – C is an arbitrary constant of integration

Definite Integrals

- We can also use integration to sum up the area under a function over some defined interval

$$F'(x) = f(x) \Rightarrow \int_{x=a}^{x=b} f(x)dx = F(b) - F(a)$$

$$\int_{x=a}^{x=b} f(x)dx = \text{area under } f(x)$$

Definite Integrals

$$\text{area under } f(x) = \int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a)$$