National University of Singapore
Microeconomic Analysis III, EC4101 (gr.2)
Tutorial 8: Game Theory
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1. The inverse demand function in an industry is given by $p = a - bq$, where p is the market price, q is the aggregate supply in the market, and a and b are positive constants. There are n firms in this industry, and each firms produces the output at a marginal cost c, where $c < a$.

   (a) Assume that $n = 2$, and firms choose output levels to maximize individual profits. Compute the Nash equilibrium of this game.

   (b) If the firms could collude by some means, could they increase their profits above those in part **(a)**? If so, can such profits be sustained?

Answer: We begin by solving the model for n firms. However, in order to do this, the following relationships need to be introduced. Define the profit function for the individual firm as

$$\Pi_i = \overbrace{pq_i}^{Revenue} - \overbrace{cq_i}^{Costs}$$

and the inverse demand function, given by

$$p = a - bQ$$

where p is the market price, q_i is the output produced by firm i, c is the cost faced by the firm of producing each unit of output, and Q is total market output (output produced by all n firms). Further, we assume that $a > 0$, $b > 0$. In order to solve for the general n firm case, it is useful to decompose *total* output in the following way:

$$Q = q_i + Q_{-i}$$

where we recall that q_i is the output produced by firm i and we let Q_{-i} denote output produced by the remaining $n - 1$ firms. In this model, a key assumption is that the market price, p, is the same across all firms. We are now well placed to solve the question.

Step 1: Substitute the inverse demand function into the profit function. This gives

$$\Pi_i = (a - bQ)q_i - cq_i.$$

Step 2: Substitute Q with $q_i + Q_{-i}$.

$$\Pi_i = (a - b(q_i + Q_{-i}))q_i - cq_i.$$

Step 3: Differentiate the profit function with respect to q_i.

$$\frac{\partial \Pi_i}{\partial q_i} = a - 2bq_i - bQ_{-i} - c = 0.$$

Step 4: Now solve for q_i. This yields

$$q_i = \frac{a-c}{2b} - \frac{Q_{-i}}{2}.$$

Summing this across **all** firms gives

$$\sum q_i \, (= Q) = \frac{n(a-c)}{2b} - \frac{(n-1)Q}{2}.$$

(Explanation: Recall, $Q = q_i + Q_{-i}$ or $Q_{-i} = Q - q_i$, so that $\sum_{i=1}^{n} Q_{-i} = nQ - Q = (n-1)Q$.)

Total output can thus be solved as:

$$Q = \frac{n(a-c)}{(n+1)b}.$$

Finally, obtain the quantity produced by firm i through dividing the above expression by n:

$$\frac{Q}{n} = q_i = \frac{\frac{n(a-c)}{(n+1)b}}{n} = \frac{a-c}{(n+1)b}.$$

Now we are ready to solve for the two firm case. All that we have to do is set $n = 2$. This tells us that the quantity that each firm will produce is given by

$$q_i = \frac{a-c}{(2+1)b} = \frac{a-c}{3b}$$

It also informs us that total industry output is given by

$$Q = \frac{2(a-c)}{(2+1)b} = \frac{2}{3}\frac{a-c}{b}$$

These are the quantities produced under Cournot equilibrium.

1. (b) To see whether firms can increase profits by colluding, maximize industry profit with respect to total industry output, Q, as follows:

$$\max_{Q} \; [(a - bQ)Q - cQ].$$

The term $(a - bQ)Q - cQ$ is merely the profit function for the **whole** industry (recall that $p = a - bQ$). Differentiating w.r.t. to Q yields

$$a - 2bQ - c = 0,$$

solving which obtain:

$$Q = \frac{a - c}{2b}.$$

For an individual firm's output share, divide profit-maximizing industry output by n:

$$q_i = \frac{a - c}{2nb}.$$

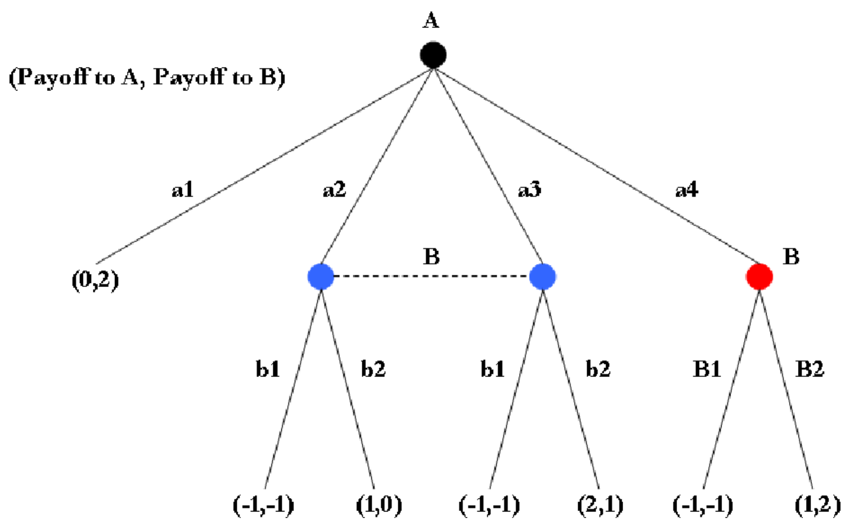
When $n = 2$,

$$q_i = \frac{a - c}{4b}.$$

This shows that optimal output is less than output when firms decide to set quantities individually (i.e. under Cournot equilibrium - compare with the solutions in part **(a)**). It turns out that $Q = \frac{a-c}{2b}$ is the amount which a firm would set if it were a monopolist - that is, if all other firms did not produce anything (note that this is equivalent to setting $n = 1$ for $q_i = \frac{a-c}{2nb}$).

As higher *output* implies *lower* profits, firms would be best to collude to increase profits - they would do this by producing *less*. Yet it is unreasonable to suppose this will happen, as there is an incentive for each firm to deviate - essentially though increasing the quantity produced, which drives down prices, and thus profits. In Cournot equilibrium there is no incentive for any firm to increase output further.

Question 2: Consider the two player game in extensive form as shown in the figure below.



a:For each player A and B, list strategies.

b:Define the Nash equilibrium. What are the Nash equilibria in the preceding game? What are the Nash outcomes?

c: Define a subgame perfect Nash equilibrium. What are subgame perfect Nash equilibria in the preceding game? What are the subgame perfect outcomes?

Answer 2: Player B has **two** *information sets.*

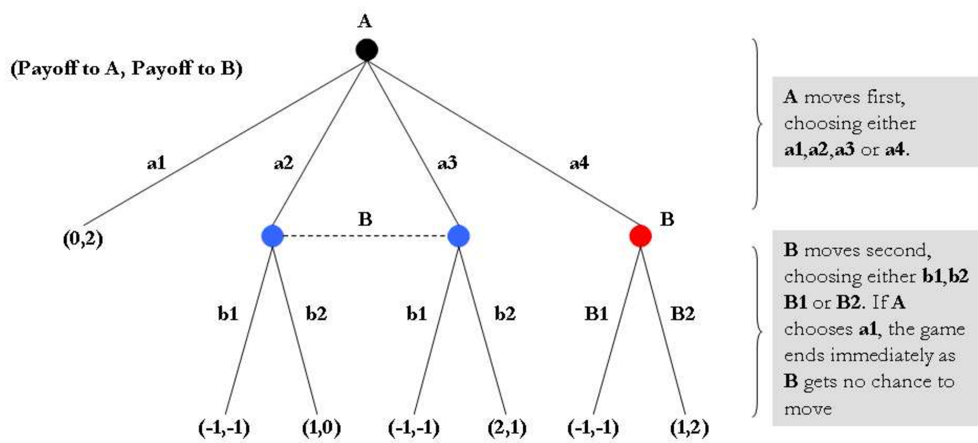
The first is characterised by the single node, which corresponds to Player A choosing action **a4**. This *information set* is also known as **singleton information set**, since it contains a single node.

The second is characterised by the two nodes, and correspond to Player A choosing either action **a2** or action **a3**. As there is more than one node in the second information set, it is **NOT** a singleton information set.

The interpretation of the game: Should Player B get the opportunity to move (this occurs if Player A *does not* choose action **a1**), she will base her decision on the following information: If Player A chooses **a4**, then Player B will learn that **a2** and **a3** were not chosen. She will know **precisely** which node has been reached as the node is a **singleton information set**. She is then faced with choosing between actions **B1** and **B2**.

However, if Player A chooses **a2** or **a3**, then Player B learns that **a4** was not chosen, but not which of **a2** or **a3** was chosen. This makes the game in the Figure given as a game of *imperfect information*. Essentially, she will *not know which blue node has been reached*, and has to choose between actions **b1** and **b2**.

Answer 2(a).



In an extensive form game, the lines out of a player's decision node(s) represent a possible *action* for that player, whereas a *strategy* is a predetermined 'programme of play' that tells a player *what* action to take in every information set the player is designated to move.

In this specific game, player A's actions are the same as his strategies – namely to play either **a1**, **a2**, **a3** or **a4**. Therefore A's actions and strategies sets are identical {**a1**, **a2**, **a3**, **a4**}.

Now consider player B. Her *actions* are given by **b1**, **b2**, **B1**, **B2**. You may alternatively group the actions according to two different information sets.

Put another way, the actions available to B are contingent on where the game ends up after A has moved: B will have a different set of actions depending on the information set he ends up at. As B moves at **two** different information sets, if she gets the opportunity to move her *strategies* can be defined as:

Strategy 1: Play **b1B1**.
Strategy 2: Play **b1B2**.
Strategy 3: Play **b2B1**.
Strategy 4: Play **b2B2**.

In other words, B has **4** strategies.

We can write these strategies as **b1B1**, **b1B2**, **b2B1** and **b2B2**. The game can now be represented in *normal form*. The rows are labeled with A's feasible strategies, and the columns with B's feasible strategies. (Note that in the matrix below, strategies are written with comma(,) separating player B's actions at two different information sets. I cannot get rid of the comma as the matrix is written in another file to which I have no access. Write the strategies in a way, with or without comma, that you feel comfortable with.)

Answer 2(b)

A **Nash equilibrium**is a strategy combination in which each player chooses a best response to the strategies chosen by the other player.

Pure-strategy Nash-Equilibria are (**a1**, **b1B1**), (**a3**, **b2B1**),(**a3**, **b2B2**) and (**a4**, (**b1**,**B2**)).
And the Nash outcomes corresponding to these equilibria are (0,2), (2,1), (2,1) and (1,2) respectively.

		B			
		b1,B1	b1,B2	b2,B1	b2,B2
A	a1	(0,2)	(0,2)	(0,2)	(0,2)
	a2	(-1,-1)	(-1,-1)	(1,0)	(1,0)
	a3	(-1,-1)	(-1,-1)	(2,1)	(2,1)
	a4	(-1,-1)	(1,2)	(-1,-1)	(1,2)

Answer 2(c)

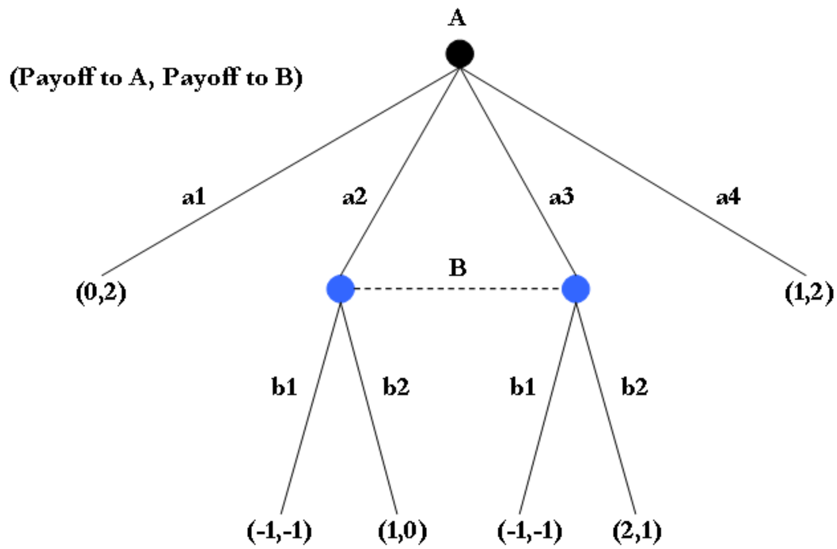
A subgame must start at an information set containing only a single node. This disqualifies an information set having two or more nodes to be the starting point of a subgame.

A subgame is a game consisting of a singleton node information set (where a specific player is designated to make a move) and all that follows the particular node – i.e., the successors to the node and the payoffs at the associated end-nodes.

Using this definition, it is easy to see that the game we are considering has two subgames:

- The whole game (trivial subgame)
- The rest of the game after the node located the end of the branch a4 (i.e., where B makes one of two moves **B1, B2**).

Returning to the non-trivial subgame, it is clear to see that for B, strategy B2 dominates B1 (as $2 > -1$): B will *always* play B2, and the extensive form game reduces to that shown in the figure above.



Therefore, all NE involving B1 can be eliminated. This means that two of the pure-strategy Nash-Equilibria highlighted in – (**a1**, **b1B1**) and (**a3**, **b2B1**) – can be eliminated. The remaining equilibria – (**a3**, **b2**,**B2**) and (**a4**, **b1**,**B2**) – constitute *subgame-perfect Nash-equilibria* (SPNE).

Subgame-perfect Nash Equilibria (SPNE) exists where players' strategies induce a Nash equilibrium in every subgame. In other words, SPNE is just a NE that induces NE in subgames. For example, when we put the extensive form game into normal from representation, we were able to identify four Nash equilibria. However, not all of these equilibria constituted NE in all of the subgames.

Since we know that B would never play B1, *all* NE involving strategies which involve B1 can be eliminated, leaving two SPNE – (**a3**, **b2B2**) and (**a4**,**b1**,**B2**).

QUESTION: **3**. Consider the following two player game. Player 1 chooses a row

(Top, Middle, or Bottom) and, simultaneously, player 2 chooses a column (Left, Middle, or Right). Each cell in the outcome matrix specifies the payoffs to players 1 and 2 respectively, for each combination of choices made by them. The pay-off matrix is represented below.

		Player 2		
		Left	**Middle**	**Right**
	Top	(7,2)	(6,5)	(3,3)
Player 1	**Middle**	(3,5)	(3,3)	(5,6)
	Bottom	(5,6)	(2,6)	(4,7)

(a) Identify all the pure-strategy Nash equilibria in the above game.

(b) Are there any mixed-strategy Nash equilibria? If so, which strategies are not used in the equilibrium randomization?

ANSWER: **3**

3. Through *iterated elimination of strictly dominated strategies* it is easy to see that for Player 2, playing Left is *strictly dominated* by Right. This is because *given her payoffs*, it is rational for Player 2 to choose Right over Left, irrespective of whether Player 1's chooses to go to the Top, Middle or Bottom(i.e. the payoff is higher in every case). **Therefore, the first column of the matrix can be eliminated and we ignore it from now on**.

Having eliminated the first column - which leaves a 3×2 matrix - it is easy to infer that for Player 1, Bottom is *strictly dominated* by Middle: irrespective of

Player 2's choice over Middle or Right. As it is always better for Player 1 to choose Middle over Bottom and accordingly, **the last row disappears**.

The payoff matrix in given in figure reduces to a so-called 'Battle of the Sexes' game.

(a). The Battle of the Sexes game is characterised by *multiple* pure strategy Nash equilibria (NE). The pure strategy NE outcomes are (**Top**, **Middle**) and (**Middle**, **Right**). Payoffs are not possible to determine.

(b). To solve for the mixed strategy equilibrium, begin by assigning probabilities to the actions available to respective players: for Player 1 (P1), denote the probability of choosing Top and Middle as p and $(1-p)$ respectively; for Player 2 (P2), denote the probability of Middle and Right as q and $(1-q)$ respectively. This is shown in Figure below. We use what is called the *payoff equating method* to determine the solution of the game. For Player 1, this involves equating the utility from choosing Top to the utility from going to see a Football match (as in the "Battle-of-the Sexes" game). Write this as

$$U(\text{Top}) = U(\text{Football}), \tag{1}$$

which can be written as

$$\underbrace{6q}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P2 chooses}\\ \text{Cinema)]}}} + \underbrace{3(1-q)}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P2 chooses}\\ \text{Football)]}}} = \underbrace{3q}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Football]} \times\\ \text{[Pr (P2 chooses}\\ \text{Cinema)]}}} + \underbrace{5(1-q)}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Football]} \times\\ \text{[Pr (P2 chooses}\\ \text{Football)]}}} \tag{2}$$

It is easy to show that solving for q yields $q = \frac{2}{5}$. Further, using the same method, it can be shown that when payoffs are equated for Player 2, $p = \frac{3}{5}$. Thus the mixed strategy NE are given by:

$$P1: \quad \begin{array}{c} \text{Chooses } \textbf{Cinema} \text{ with probability } p = \frac{3}{5} \\ \text{Chooses } \textbf{Football} \text{ with probability } (1-p) = \frac{2}{5} \end{array} \tag{3}$$

and

$$P2: \quad \begin{array}{c} \text{Chooses } \textbf{Cinema} \text{ with probability } q = \frac{2}{5} \\ \text{Chooses } \textbf{Football} \text{ with probability } (1-q) = \frac{3}{5} \end{array} \tag{4}$$

Using these probabilities, we are in a position to calculate players' payoffs. For Player 1, we just substitute the numerical values of p and q (obtained using the *payoff equating method*) back into the LHS or the RHS of our expression using the payoff equating method shown above. Player 1 gets

$$\overbrace{\underbrace{6q}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P2 chooses}\\ \text{Cinema)]}}} + \underbrace{3(1-q)}_{\substack{\text{[Payoff to P1 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P2 chooses}\\ \text{Football)]}}}}^{U(\text{Cinema})} = 6 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{21}{5} \tag{5}$$

and Player 2 gets

$$\overbrace{\underbrace{5p}_{\substack{\text{[Payoff to P2 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P1 chooses}\\ \text{Cinema)]}}} + \underbrace{3(1-p)}_{\substack{\text{[Payoff to P2 from}\\ \text{choosing Cinema]} \times\\ \text{[Pr (P1 chooses}\\ \text{Football)]}}}}^{U(\text{Cinema})} = 5 \cdot \frac{3}{5} + 3 \cdot \frac{2}{5} = \frac{21}{5} \tag{6}$$

The payoffs are the same. This makes intuitive sense, as the reduced game is symmetric.