EC4101
Microeconomics Analysis III
(L2)

Topic 1

Consumer Theory

Topic 1: Consumer Theory

- Preferences and utility (ch. 3)
- Utility maximization and choice (ch. 4)
- Income and substitution effect (ch. 5)
- Demand relationships among goods (ch. 6)

Consumer Problem

- Choose the best possible consumption bundle

Learning Objectives

1. Preference and Utility
   – How they are related?
   – Representation by indifference curve
   – Marginal rate of substitution
2. Consumer's optimization
   – Demand function
   – indirect utility function
   – Expenditure function

Preference and Utility

Reading

- Main Reading
  - Chapter 3 of Textbook
- Technical reference
  - Chapter 1 of Jehle/Reny

Preference

- Given a set of situations, preference satisfies two conditions:
  - ability to compare (completeness)
  - internal consistency (transitivity)

Completeness

- Consider any two situations A and B
  - An individual can always specify exactly one of these possibilities:
    - A is preferred to B
    - B is preferred to A
    - A and B are equally attractive

Transitivity

- Consider any three situations A, B, C
  - If A is preferred to B, and B is preferred to C, then A is preferred to C
  - Assumes that the individual's choices are internally consistent

Technical Note: Preference Relation

- Consumption set: X
- Binary relation defined on X:
  - It is ordered pair (hence it is subset of X^2)
  - we write $(x_1,x_2)\in R$ as x_1Rx_2
  - x_1Rx_2 : x_1 at least as good as x_1
- Completeness: either xRy or yRx or both
- Transitivity: xRy and yRz implies xRz
- Reference: Jehel and Reny Chpater 1

Utility

- Utility represents preference
  - If A is preferred to B
  - Then the utility assigned to A exceeds the utility assigned to B: $U(\mathrm{A}) > U(\mathrm{B})$

Utility

- Not always possible
- Example:
  - lexicographic preference
  - No way to assign utility
- Preference $\neq$ Utility

Continuity

- Technical Condition for existence of utility
  – If A is preferred to B, then situations suitably "close to" A must also be preferred to B
  – Used to analyze individuals' responses to relatively small changes in income and prices

Technical Note: Existence of Utility

- Proposition:
  - If a preference (=complete + transitive) is (1) continuous and (2) strictly monotonic (more is strictly better), then a utility function <u>exists</u>.
- Proof:
  - See Jehle and Reny (P.14-17)

Utility

- Utility represents preference
- Preference is a (rational) comparison
  - absolute level doesn't matter
  - only relative matters!
  - unique only up to an order-preserving transformation

Ordinal Utility

- ordinal in nature
  - will talk about cardinal utility in topic 4
- can’t say how much one situation is different from others
  - You can say how apple compare to orange
  - You can’t say how many times apple is better/worse than orange
- can’t compare between different people

Utility

- Utility from consumption of goods
  - Assume - an individual must choose among consumption goods $x_1, x_2, \ldots, x_n$
  - Show his rankings using a utility function of the form: utility = $U(x_1, x_2, \ldots, x_n;$ other things)
  - Often "other things" are held constant, so

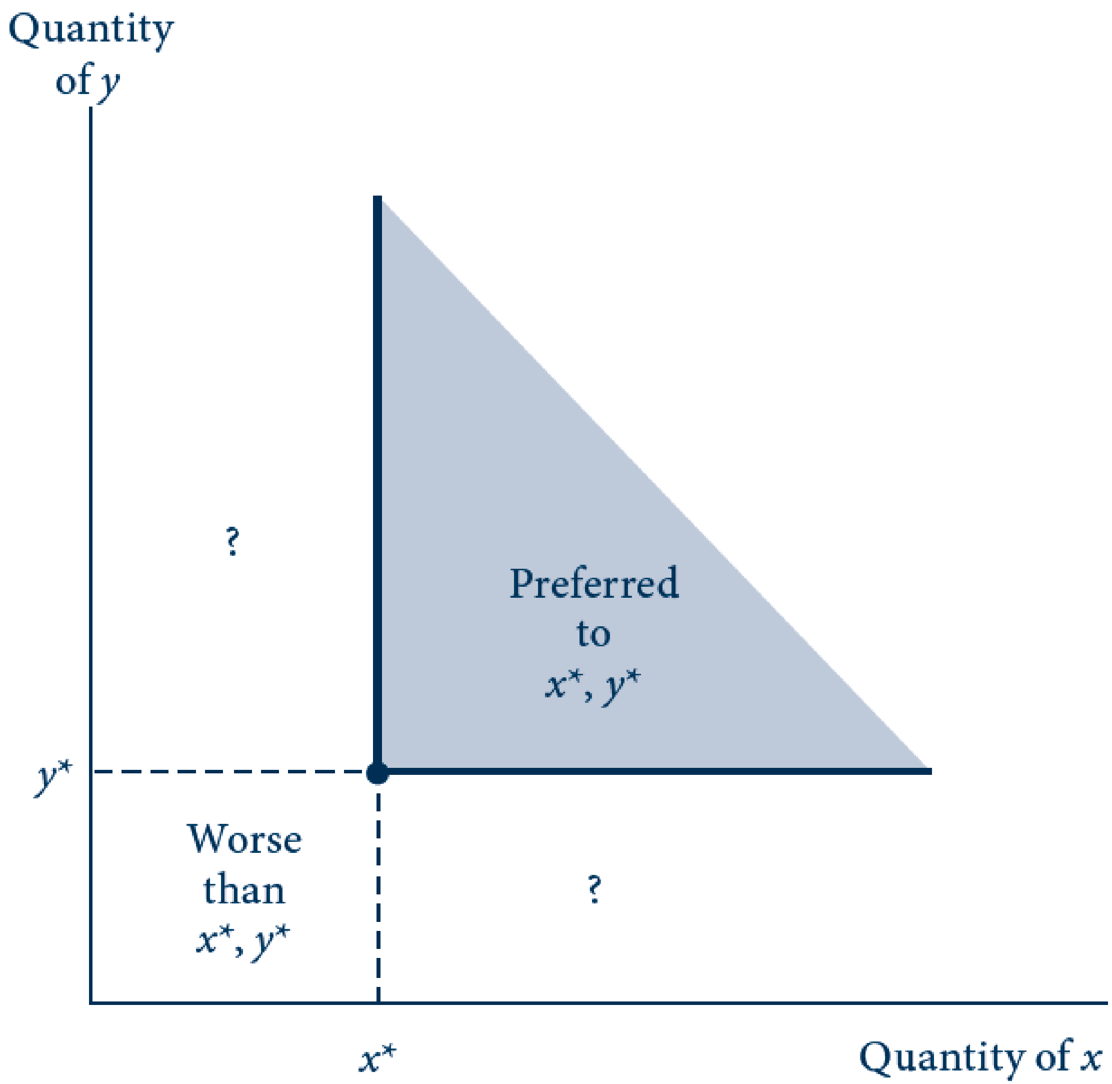
    utility = $U(x_1, x_2, \ldots, x_n)$
  - For two goods, x and y: utility = $U(x,y)$

Utility

- Arguments of utility functions
  - U(W) = utility an individual receives from real wealth (W)
  - U(c,h) = utility from consumption (c) and leisure (h)
  - $U(c_1,c_2)$ = utility from consumption in two different periods
- Two-good utility function U(x,y)
  - More of any particular x_i during some period is preferred to less

FIGURE 3.1

More of a Good Is Preferred to Less



The shaded area represents those combinations of x and y that are unambiguously preferred to the combination x*, y*. Ceteris paribus, individuals prefer more of any good rather than less. Combinations identified by “?” involve ambiguous changes in welfare because they contain more of one good and less of the other.

Indifference curve

- represent preference (utility)
- showing bundles having same utility (indifference)
  - In mathematics, level curve
  - In geography, contour

Example: Indifference graph

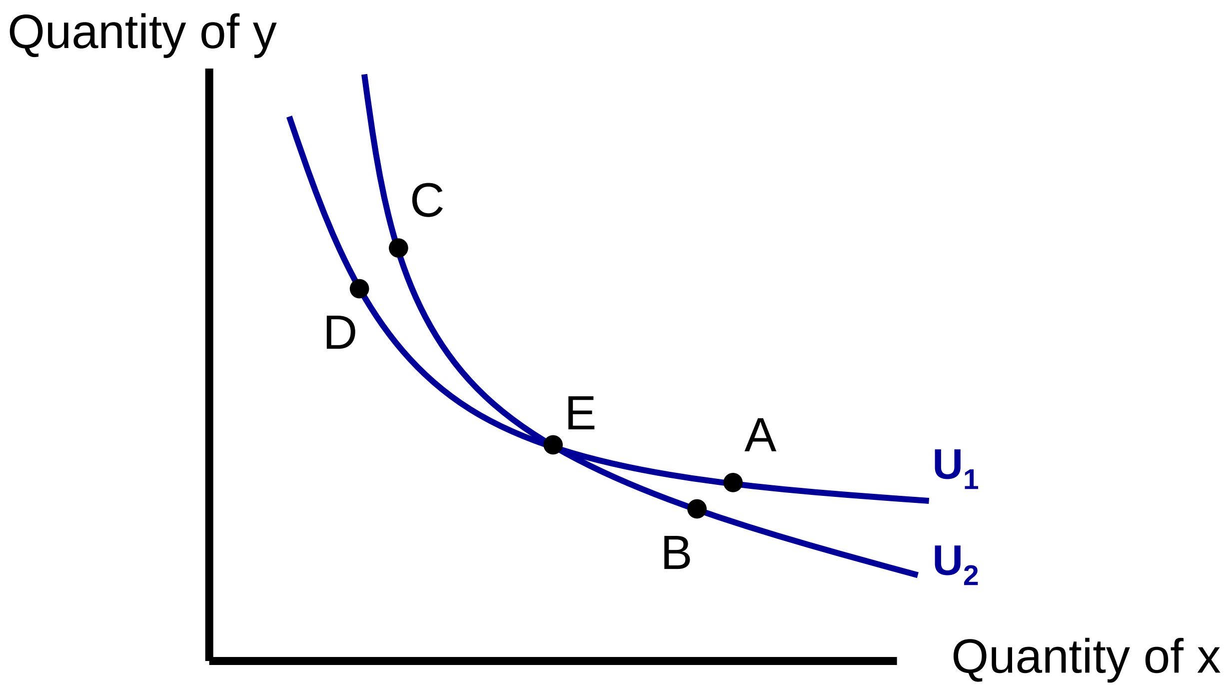
- My turn: $U(x,y) = x+y$

- Your turn: $U(x,y) = x+2y$

Indifference curve

- Cannot intersect
- Why?

Intersecting Indifference Curves Imply Inconsistent Preferences



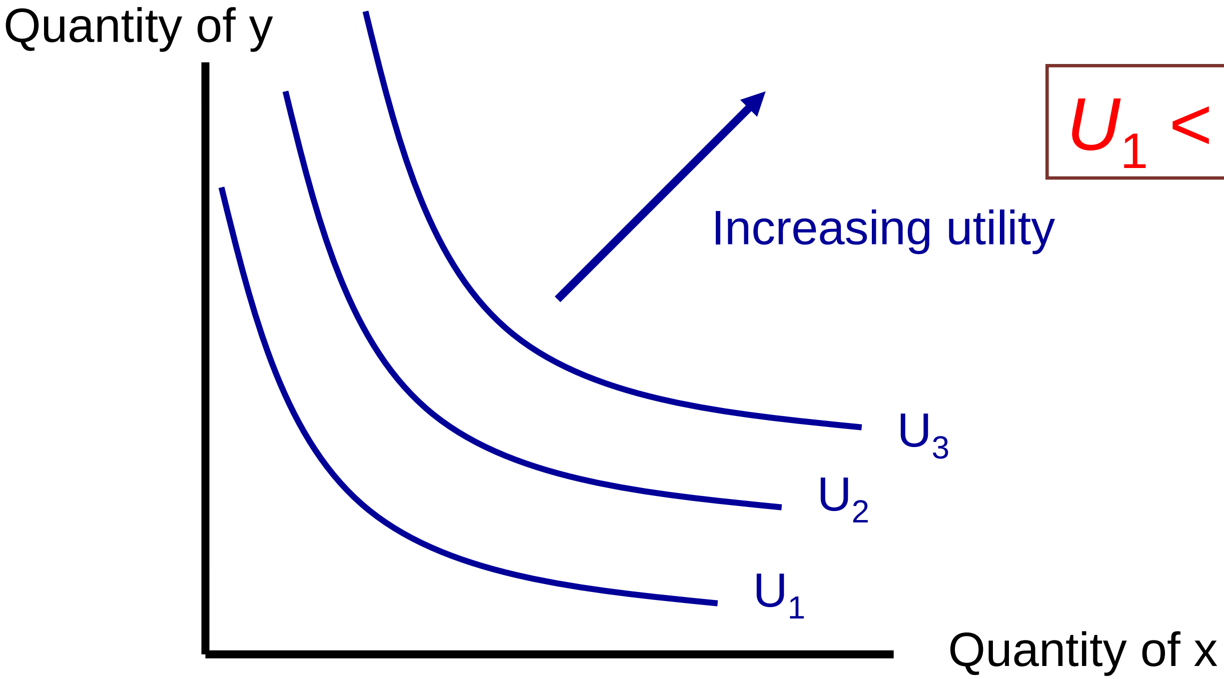
Combinations A and D lie on the same indifference curve and therefore are equally desirable. But the axiom of transitivity can be used to show that A is preferred to D. Hence intersecting indifference curves are not consistent with rational preferences.

Monotonic Preference

- More is better
- Formally: (x_1,x_2) is preferred to (y_1,y_2) if $x_1 \geq y_1$ and $x_2 \geq y_2$.
- Then, northeast direction is higher utility

FIGURE 3.3

There Are Infinitely Many Indifference Curves in the x–y Plane



$U_1 < U_2 < U_3$

There is an indifference curve passing through each point in the x–y plane. Each of these curves records combinations of x and y from which the individual receives a certain level of satisfaction. Movements in a northeast direction represent movements to higher levels of satisfaction.

Convex Preference

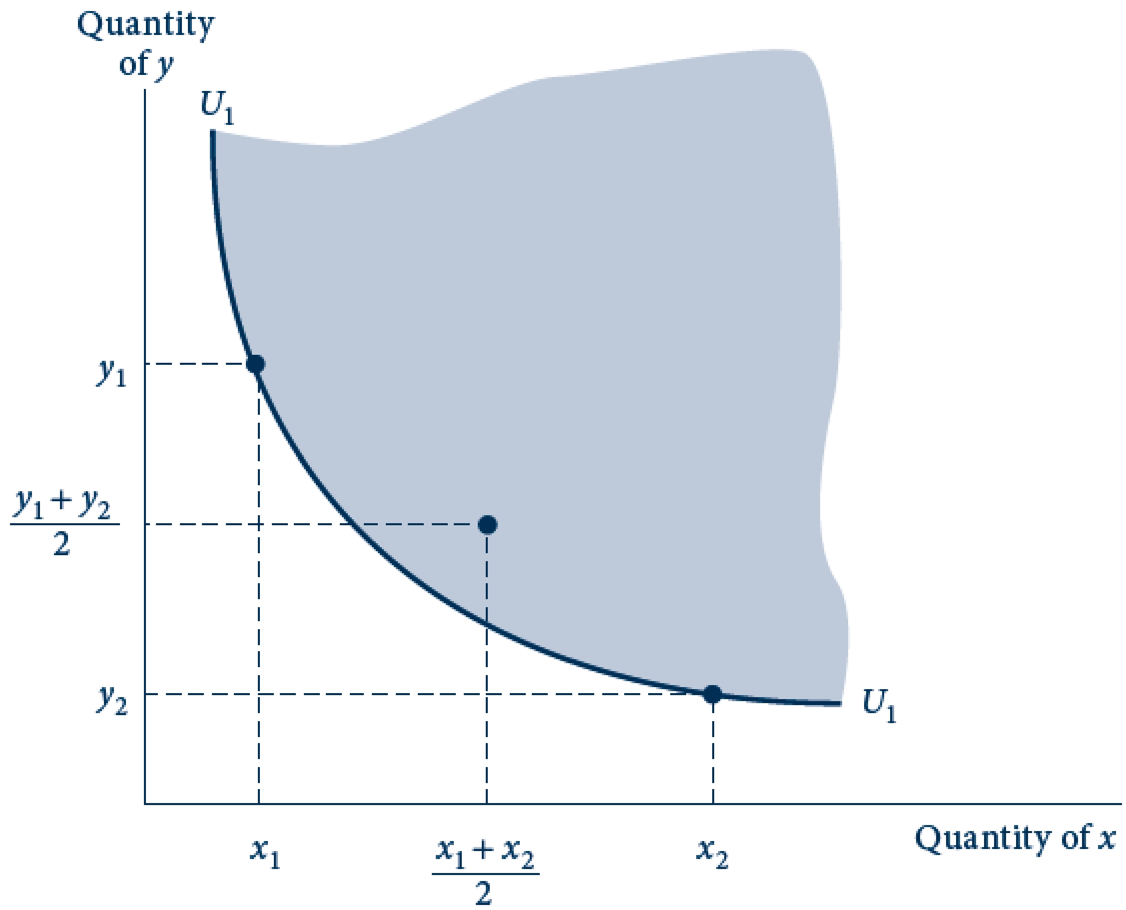
- Convex combination is no worse
  - Canteen: mixing dishes are more expensive
- Why convex? Draw indifference curves

  A set of points is convex if

  – Any two points can be joined by a straight line that is contained completely within the set
- Related to marginal rate of substitution (MRS)
  – Recall MRS?

FIGURE 3.6

Balanced Bundles of Goods Are Preferred to Extreme Bundles



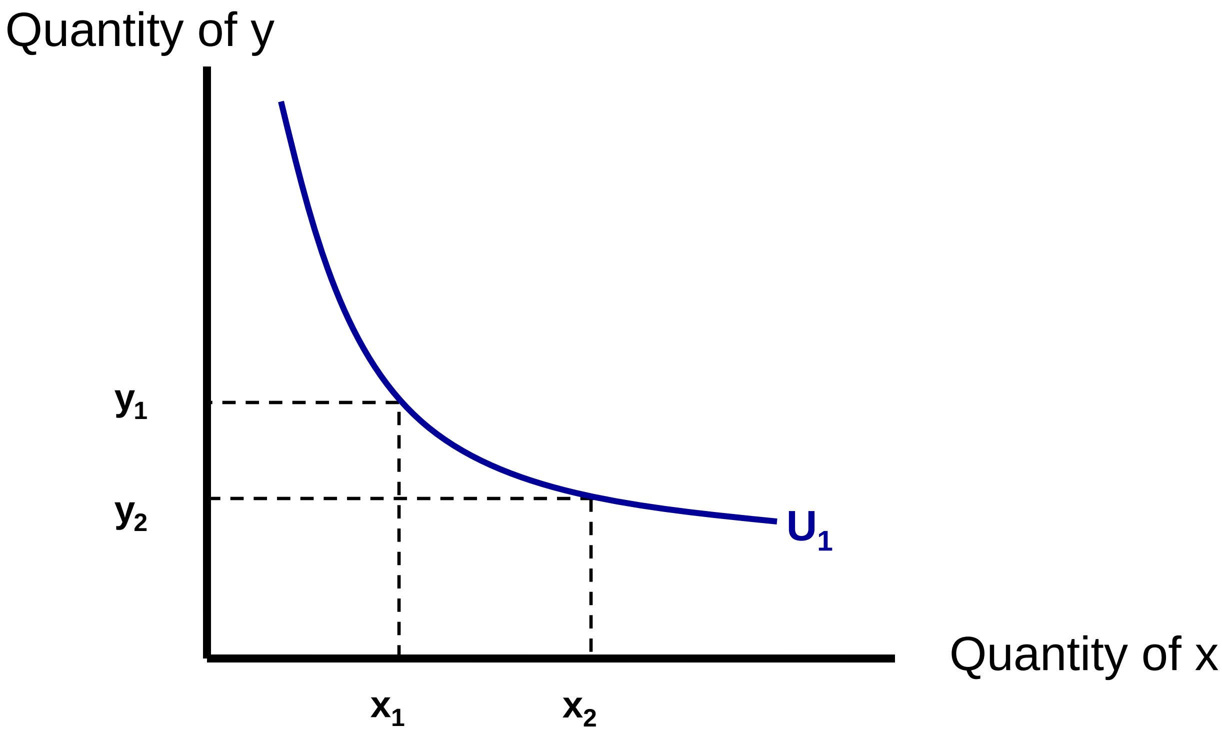
If indifference curves are convex (if they obey the assumption of a diminishing MRS), then the line joining any two points that are indifferent will contain points preferred to either of the initial combinations. Intuitively, balanced bundles are preferred to unbalanced ones.

Trades and Substitution

- Marginal rate of substitution, MRS
  - The negative of the slope of an indifference curve (U_1) at some point
    - Marginal rate of substitution at that point
  - MRS changes as *x* and *y* change
    - Reflects the individual's willingness to trade *y* for *x*

$$MRS = -\left.\frac{dy}{dx}\right|_{U=U_1}$$

A Single Indifference Curve



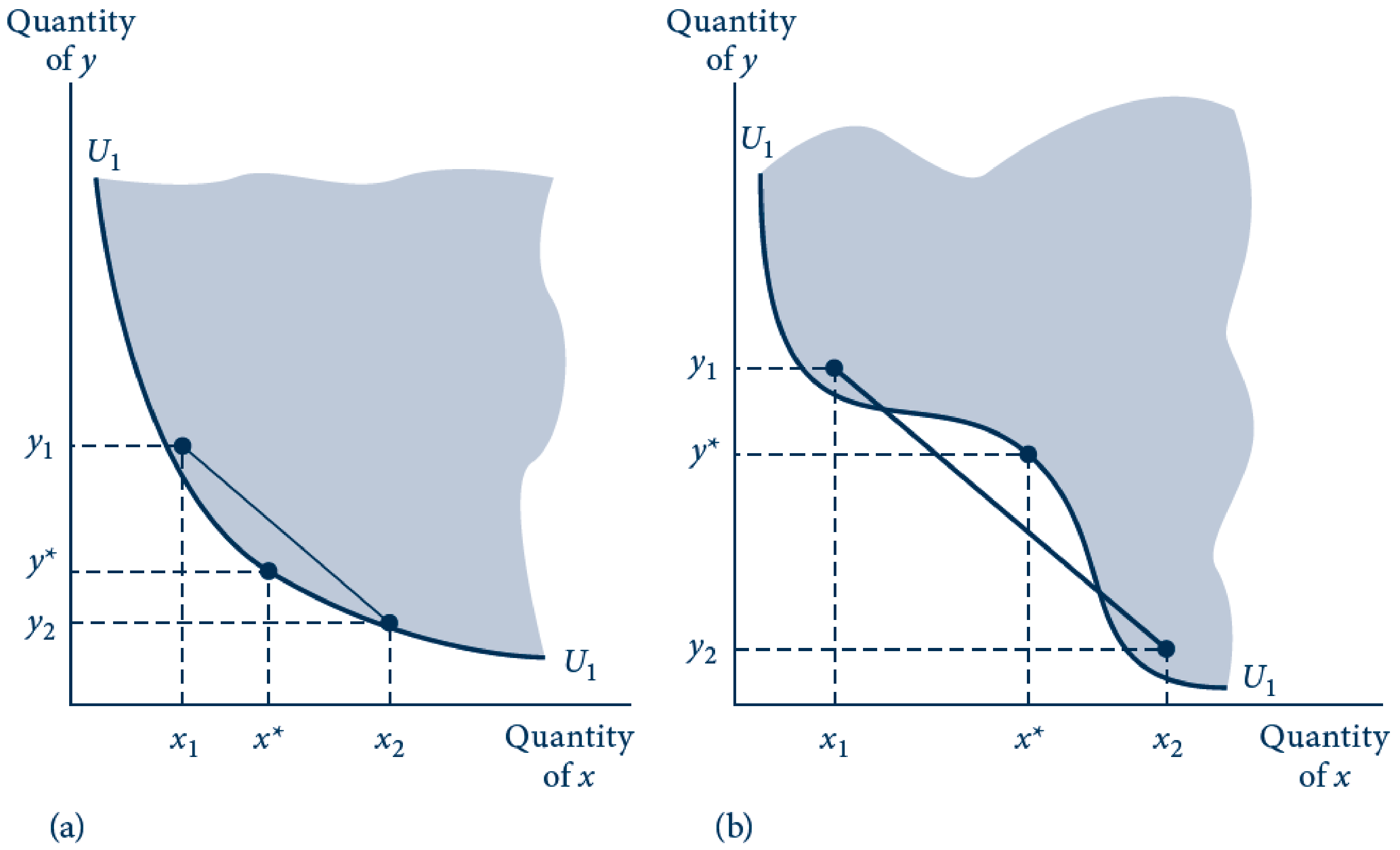
The curve U_1 represents those combinations of x and y from which the individual derives the same utility. The slope of this curve represents the rate at which the individual is willing to trade x for y while remaining equally well off. This slope (or, more properly, the negative of the slope) is termed <u>the marginal rate of substitution</u>. In the figure, the indifference curve is drawn on the assumption of a diminishing marginal rate of substitution.

Technical Note: MRS and Convexity

- Diminishing MRS ⇔ Convex "Upper set"
- Convex "Upper set" ⇔ quasiconcave utility
  – Quasiconave: convex combination is no less than the worst of original bundle

FIGURE 3.5

The Notion of Convexity as an Alternative Definition of a Diminishing MRS



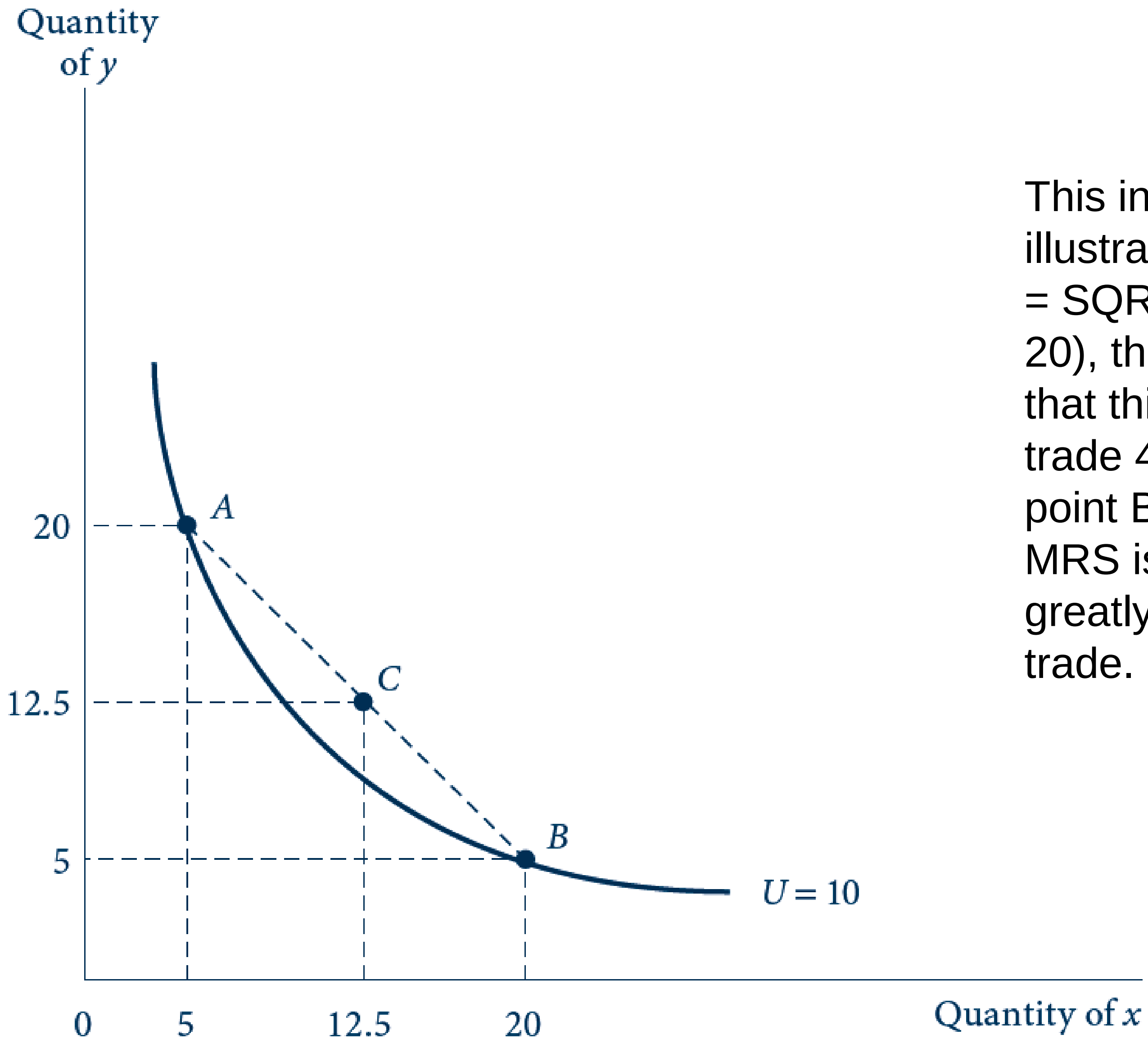
In (a) the indifference curve is convex (any line joining two points above U_1 is also above U_1). In (b) this is not the case, and the curve shown here does not everywhere have a diminishing MRS.

EXAMPLE 3.1 Utility and the MRS

- A person's ranking of hamburgers (y) and soft drinks (x)
  - Utility = SQRT(x·y)
- An indifference curve for this function
  - Identify that set of combinations of x and y for which utility has the same value
  - Utility = 10, so 100=x·y, therefore y=100/x
- MRS = -dy/dx(along U_1)=$100/x^2$
  - As *x* rises, *MRS* falls
    - When *x* = 5, *MRS* = 4
    - When *x* = 20, *MRS* = 0.25

FIGURE 3.7

Indifference Curve for Utility=SQRT(x·y)



This indifference curve illustrates the function 10 = U = SQRT(x·y) . At point A (5, 20), the MRS is 4, implying that this person is willing to trade 4y for an additional x. At point B (20, 5), however, the MRS is 0.25, implying a greatly reduced willingness to trade.

The Mathematics of Indifference Curves

- An individual – consumes x and y
  - Utility = U(x,y)
  - Specific level of utility, k: U(x,y)=k
  - Trade-offs: the rate at which x can be traded for y
    - Is given by the negative of the ratio of the "marginal utility" of good x to that of good y

$$MRS = -\left.\frac{dy}{dx}\right|_{U(x,y)=k} = \frac{U_x}{U_y}$$

EXAMPLE 3.2 Showing Convexity of Indifference Curves

1. $U(x, y) = \sqrt{x \cdot y}$

Let $U^*(x, y) = \ln[U(x, y)] = 0.5 \ln x + 0.5 \ln y$

$$MRS = \frac{\partial U^* / \partial x}{\partial U^* / \partial y} = \frac{y}{x}$$

- MRS is diminishing as x increases and y decreases
- Therefore, the indifference curves are convex

EXAMPLE 3.2 Showing Convexity of Indifference Curves

$$2.\ U(x,y) = x + xy + y$$

$$MRS = \frac{\partial U / \partial x}{\partial U / \partial y} = \frac{1+y}{1+x}$$

- MRS is diminishing as x increases and y decreases
- Therefore, the indifference curves are convex

EXAMPLE 3.2 Showing Convexity of Indifference Curves

$$3.\ U(x,y)=\sqrt{x^2+y^2}$$

$$\text{Let}\quad U^*(x,y)=[U(x,y)]^2=x^2+y^2$$

$$MRS=\frac{\partial U^*/\partial x}{\partial U^*/\partial y}=\frac{x}{y}$$

- As x increases and y decreases, the MRS increases!
- The indifference curves are concave, not convex
- This is not a quasi-concave function

Utility Functions for Specific Preferences

- Cobb-Douglas Utility

  utility = $U(x,y) = x^{\alpha}y^{\beta}$

  - Where α and β are positive constants
  - The relative sizes of α and β indicate the relative importance of the goods
  - Normalize so that $\alpha + \beta = 1$

    $U(x,y) = x^{\delta}y^{1-\delta}$

    - Where $\delta=\alpha/(\alpha+\beta)$ and $1-\delta=\beta/(\alpha+\beta)$

Utility Functions for Specific Preferences

- Perfect substitutes
  - Linear indifference curves

    utility = $U(x,y) = \alpha x + \beta y$
  - Where α and β are positive constants
  - The MRS will be constant along the indifference curves

Utility Functions for Specific Preferences

- Perfect complements
  - L-shaped indifference curves

    utility = $U(x,y) = \min(\alpha x, \beta y)$
  - Where α and β are positive parameters

Utility Functions for Specific Preferences

- CES Utility (constant elasticity of substitution)

  utility = $U(x,y) = x^{\delta}/\delta + y^{\delta}/\delta$

  when $\delta \leq 1$, $\delta \neq 0$ and

  utility = $U(x,y) = \ln x + \ln y$

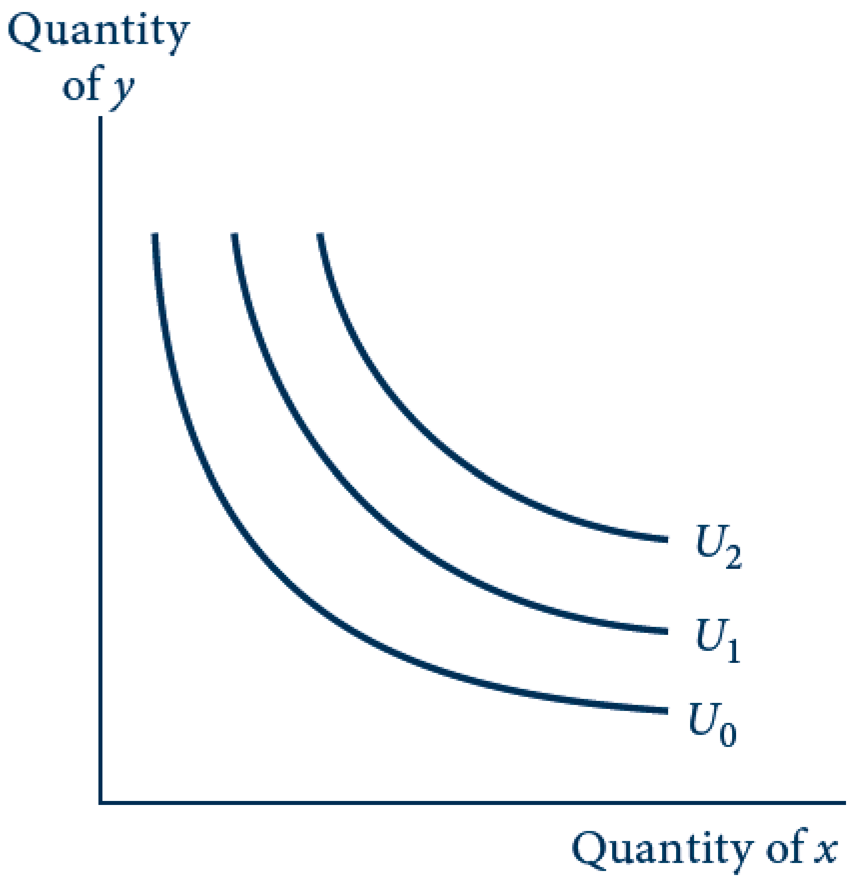
  when $\delta = 0$
- Perfect substitutes $\Rightarrow \delta = 1$
- Cobb-Douglas $\Rightarrow \delta = 0$
- Perfect complements $\Rightarrow \delta = -\infty$

Utility Functions for Specific Preferences

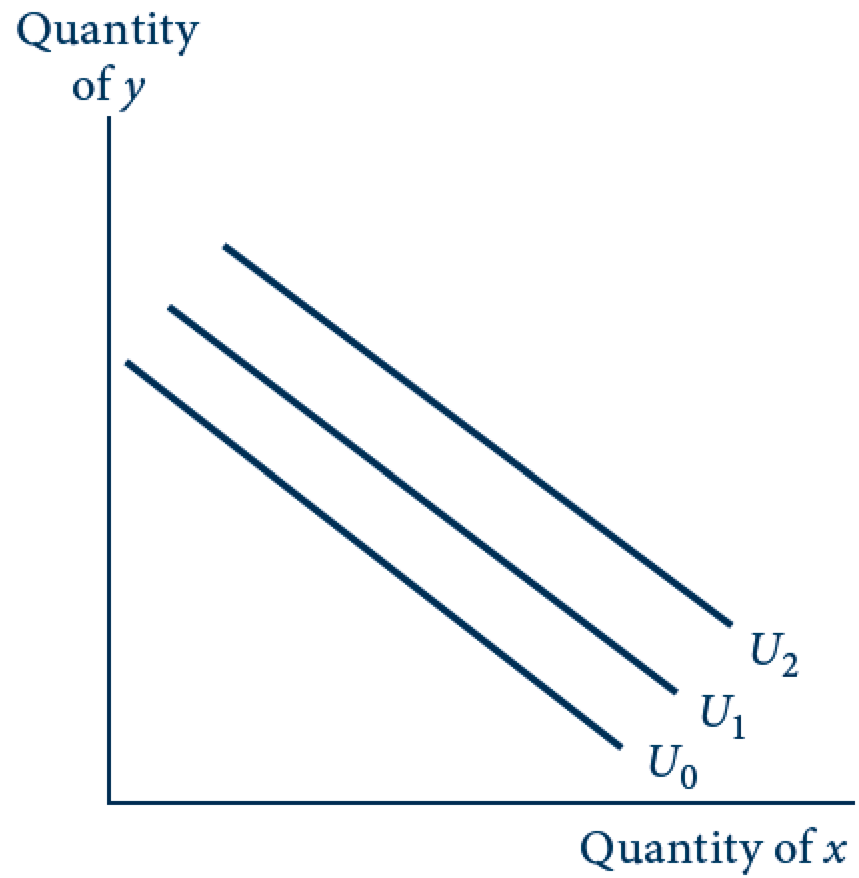
- The elasticity of substitution, σ
  - CES utility $\Rightarrow \sigma = 1/(1 - \delta)$
  - Perfect substitutes $\Rightarrow \sigma = \infty$
  - Perfect complements $\Rightarrow \sigma = 0$

FIGURE 3.8 a, b

Examples of Utility Functions



(a) Cobb-Douglas

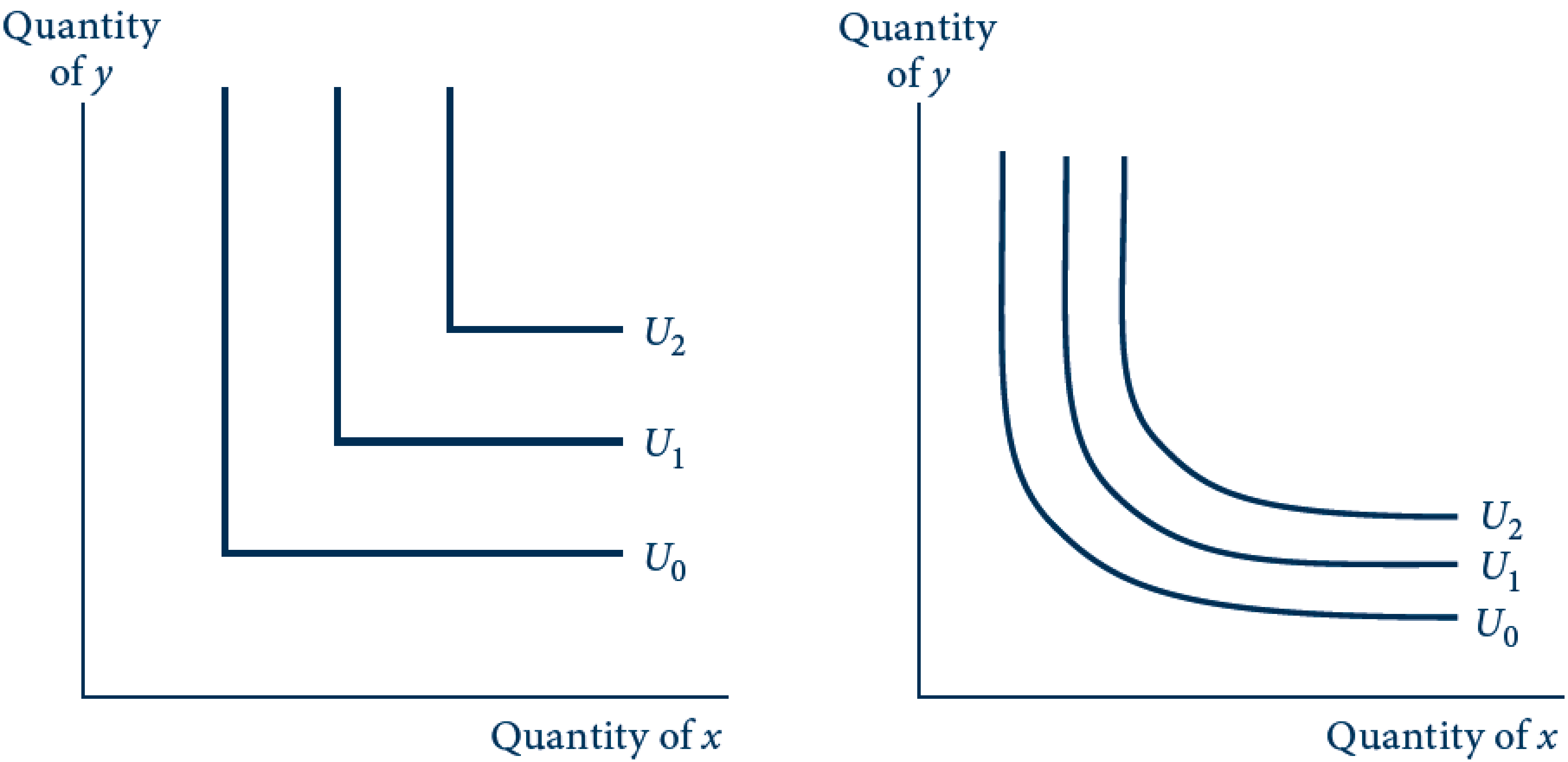


(b) Perfect substitutes

The four indifference curve maps illustrate alternative degrees of substitutability of x for y. The Cobb–Douglas and constant elasticity of substitution (CES) functions (drawn here for relatively low substitutability) fall between the extremes of perfect substitution (b) and no substitution (c).

FIGURE 3.8 c, d

Examples of Utility Functions



(c) Perfect complements

(d) CES

The four indifference curve maps illustrate alternative degrees of substitutability of x for y. The Cobb–Douglas and constant elasticity of substitution (CES) functions (drawn here for relatively low substitutability) fall between the extremes of perfect substitution (b) and no substitution (c).

EXAMPLE 3.3 Homothetic Preferences

- Utility function is homothetic
  - If the *MRS* depends only on the ratio of the amounts of the two goods
- Perfect substitutes
  - *MRS* is the same at every point
- Perfect complements
  - *MRS* = ∞ if $y/x > \alpha/\beta$
  - *MRS* is undefined if $y/x = \alpha/\beta$
  - *MRS* = 0 if $y/x < \alpha/\beta$

- General Cobb-Douglas function
  - The *MRS* depends only on the ratio y/x

$$MRS = \frac{\partial U / \partial x}{\partial U / \partial y} = \frac{\alpha x^{\alpha-1} y^{\beta}}{\beta x^{\alpha} y^{\beta-1}} = \frac{\alpha}{\beta} \cdot \frac{y}{x}$$

Nonhomothetic Preferences

- Some utility functions do not exhibit homothetic preferences

utility = $U(x,y) = x + \ln y$

- Good y exhibits diminishing marginal utility, but good x does not
- The MRS diminishes as the chosen quantity of y decreases, but it is independent of the quantity of x consumed

$$MRS = \frac{\partial U / \partial x}{\partial U / \partial y} = \frac{1}{1/y} = y$$

The Many-Good Case

- Suppose utility is a function of *n* goods given by

  utility = $U(x_1, x_2, \ldots, x_n)$

- $U(x_1, x_2, \ldots, x_n) = k$
  - Defines an indifference surface in n dimensions
    - All those combinations of the n goods that yield the same level of utility (Convex surface)
  - Quasi-concave

The Many-Good Case

- MRS with many goods

$$MRS = -\left.\frac{dx_2}{dx_1}\right|_{U(x_1,x_2,\ldots,x_n)=k} = \frac{U_{x_1}(x_1,x_2,\ldots,x_n)}{U_{x_2}(x_1,x_2,\ldots,x_n)}$$

Special Preferences

- The utility function
  - General concept
  - Can be adapted to a large number of special circumstances
- Aspects of preferences that economists have tried to model

  (1) threshold effects

  (2) quality

  (3) habits and addiction

  (4) second-party preferences

Threshold effects

- People may be “set in their ways”
  - May require a rather large change in circumstances to change what they do
  - Assume individuals make decisions as though they faced thresholds of preference
- Bundle A might be chosen over B only when: $U(A) > U(B) + \varepsilon$
  - Where ε is the threshold that must be overcome

Quality

- Many consumption items differ in quality
  - Focus on quality as a direct item of choice
- Utility = $U(q,Q)$
  - q is the quantity consumed
  - Q is the quality of that consumption
- Utility = $U[q,a_1(q),a_2(q)]$
  - Good q provides a well-defined set of attributes of goods (a)
    - Assumes that those attributes provide utility

Habits and addiction

- Habits
  - Are formed when individuals discover they enjoy using a commodity in one period
  - And this increases their consumption in subsequent periods
- Addiction
  - An extreme case of habits
  - Past consumption significantly increases the utility of present consumption

Habits and addiction

- Utility = $U(x_t, y_t, s_t)$
  - Utility in period t depends on
    - Consumption in period t and the total of all previous consumption

$$s_t = \sum_{i=1}^{\infty} x_{t-i}$$

- Utility = $U(x_t^*, y_t)$
  - x_t^* is a function of
    - Current consumption (x_t)
    - And consumption in the previous period (x_{t-1})

Second-party preferences

- Second-party preferences
  - Can be incorporated into the utility function of person i
- Utility = $U_i(x_i,y_i,U_j)$
  - Where U_j is the utility of someone else
- If $\partial U_i/\partial U_j>0$
  - This person will engage in altruistic behavior

Second-party preferences

- If $\partial U_i/\partial U_j<0$
  - This person will demonstrate the malevolent behavior associated with envy
- If $\partial U_i/\partial U_j=0$
  - The usual case
  - Middle ground between these alternative preference types

Utility Maximization

Reading

- Main Reading
  - Chapter 4 of Textbook
- Technical reference
  - Chapter 1 of Jehle/Reny

Recall: Consumer Problem

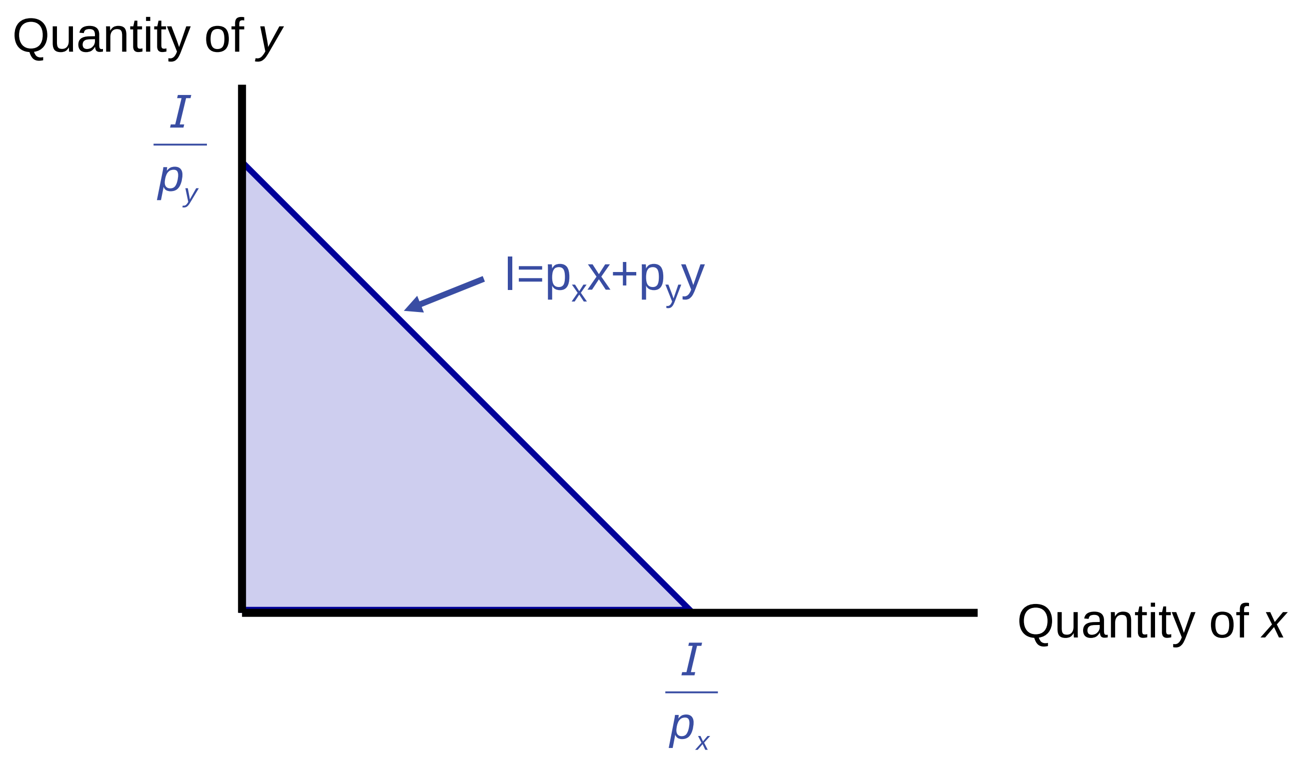
- Choose the best possible consumption bundle
- What is the best?
  - highest utility
- What is possibility?
  - Budget constraint
- Maximize utility subject to budget constraint

Two-good Case

- Denote
  - p_x - price of good x
  - p_y - price of good y
  - I - budget
- Budget constraint: $p_x x + p_y y \leq I$
  - Slope = $-p_x/p_y$

FIGURE 4.1

The Individual's Budget Constraint for Two Goods



Those combinations of x and y that the individual can afford are shown in the shaded triangle. If, as we usually assume, the individual prefers more rather than less of every good, the outer boundary of this triangle is the relevant constraint where all the available funds are spent either on x or on y. The slope of this straight-line boundary is given by $-p_x/p_y$.

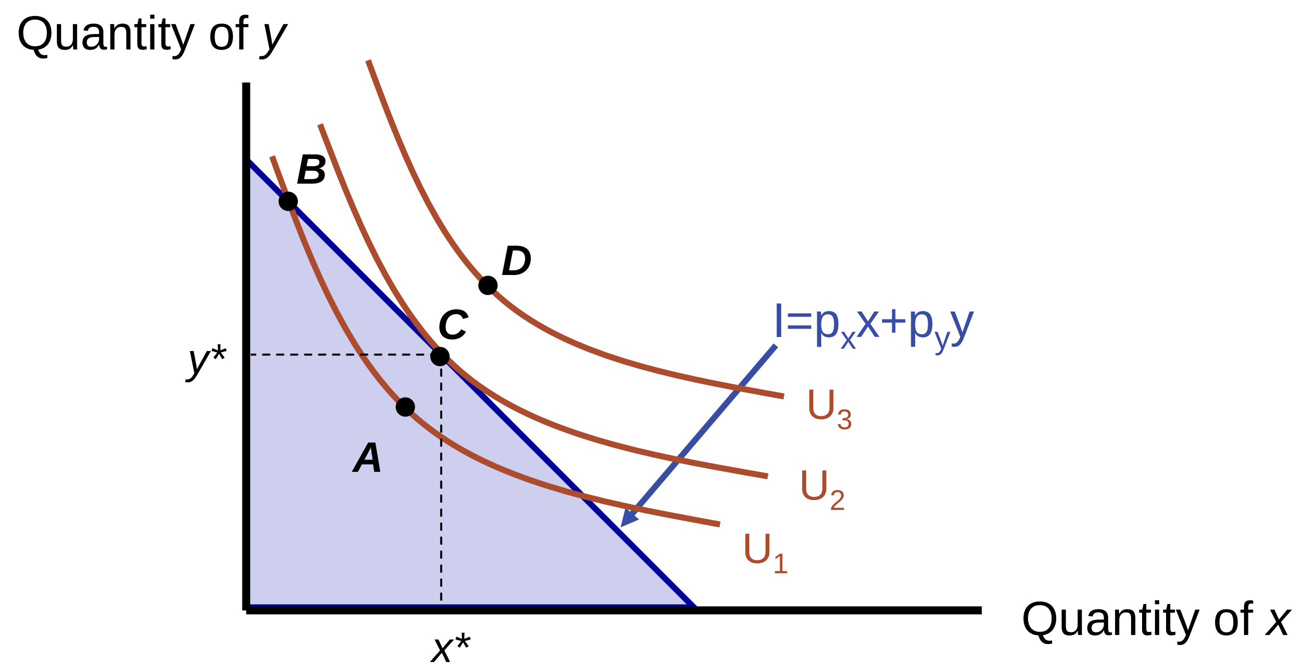
The Two-Good Case

- First-order conditions for a maximum
  - Point of tangency between the budget constraint and the indifference curve:

slope of budget constraint=slope of indifference curve

$$\frac{p_x}{p_y} = -\left.\frac{dy}{dx}\right|_{\text{U=constant}} = MRS(\text{of } x \text{ for } y)$$

A Graphical Demonstration of Utility Maximization



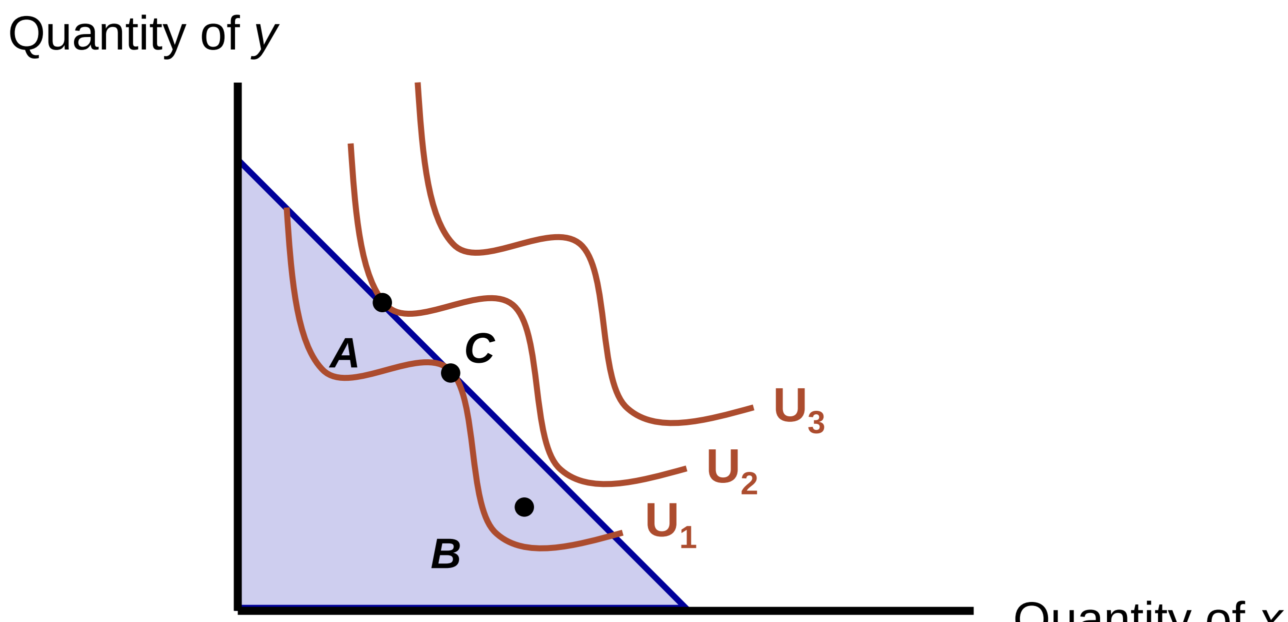
Point C represents the highest utility level that can be reached by the individual, given the budget constraint. Therefore, the combination x*,y* is the rational way for the individual to allocate purchasing power. Only for this combination of goods will two conditions hold: All available funds will be spent, and the individual's psychic rate of trade-off (MRS) will be equal to the rate at which the goods can be traded in the market (p_x/p_y).

The Two-Good Case

- The tangency rule
  - Is <u>necessary but not sufficient</u> unless we assume that *MRS* is diminishing
    - If *MRS* is diminishing, then indifference curves are strictly convex
    - If *MRS* is not diminishing, we must check second-order conditions to ensure that we are at a maximum

FIGURE 4.3

Example of an Indifference Curve Map for Which the Tangency Condition Does Not Ensure a Maximum



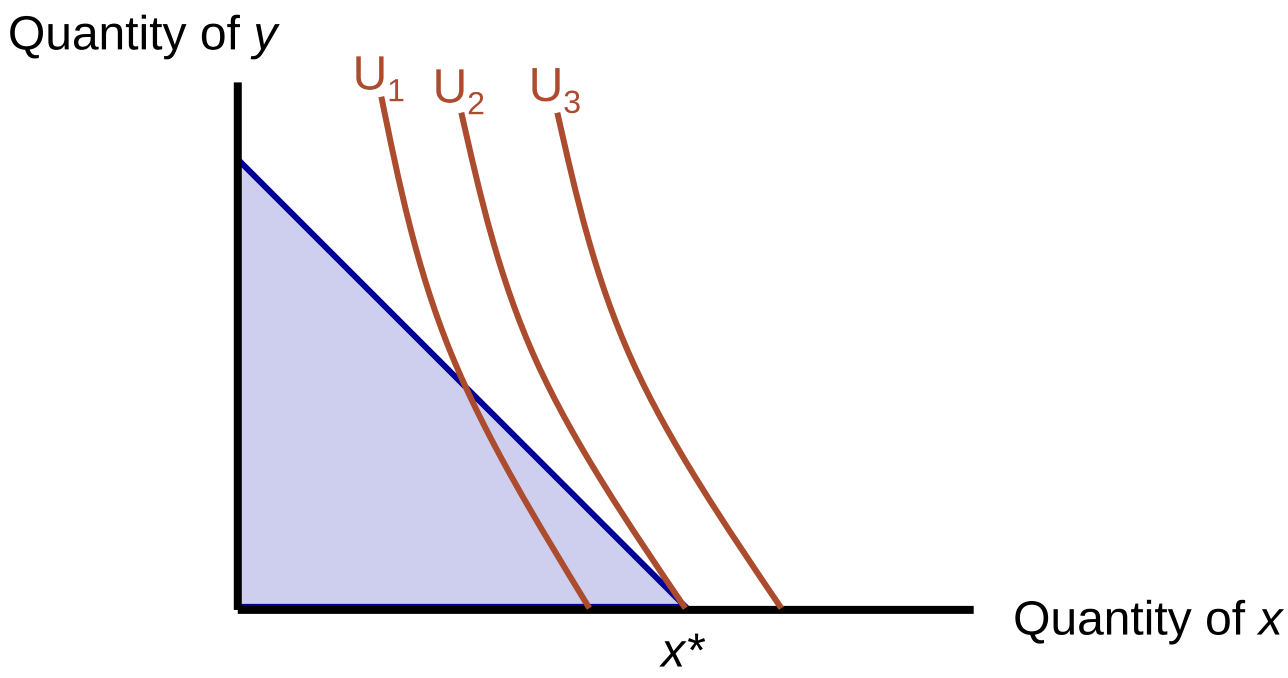
If indifference curves do not obey the assumption of a diminishing MRS, not all points of tangency (points for which MRS = p_x/p_y) may truly be points of maximum utility. In this example, tangency point C is inferior to many other points that can also be purchased with the available funds. In order that the necessary conditions for a maximum (i.e., the tangency conditions) also be sufficient, one usually assumes that the MRS is diminishing; that is, the utility function is strictly quasi-concave.

The Two-Good Case

- Corner solutions
  - Individuals may maximize utility by choosing to consume only one of the goods
  - At the optimal point the budget constraint is flatter than the indifference curve
    - The rate at which x can be traded for y in the market is lower than the MRS

FIGURE 4.4

Corner Solution for Utility Maximization



With the preferences represented by this set of indifference curves, utility maximization occurs at E, where 0 amounts of good y are consumed. The first-order conditions for a maximum must be modified somewhat to accommodate this possibility.

The n-Good Case

- The individual's objective is to maximize

$$\text{utility} = U(x_1, x_2, \ldots, x_n)$$

subject to the budget constraint

$$I = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

- Set up the Lagrangian:

$$\mathscr{L} = U(x_1, x_2, \ldots, x_n) + \lambda(I - p_1x_1 - p_2x_2 - \ldots - p_nx_n)$$

The n-Good Case

- First-order conditions for an interior maximum

$$\partial \mathscr{L}/\partial x_1 = \partial U/\partial x_1 - \lambda p_1 = 0$$

$$\partial \mathscr{L}/\partial x_2 = \partial U/\partial x_2 - \lambda p_2 = 0$$

$$\dots$$

$$\partial \mathscr{L}/\partial x_n = \partial U/\partial x_n - \lambda p_n = 0$$

$$\partial \mathscr{L}/\partial \lambda = I - p_1 x_1 - p_2 x_2 - \dots - p_n x_n = 0$$

The n-Good Case

- Implications of first-order conditions
  - For any two goods, x_i and y_j:

$$\frac{\partial U / \partial x_i}{\partial U / \partial x_j} = \frac{p_i}{p_j} = MRS\ (x_i \text{ for } x_j)$$

The n-Good Case

- Interpreting the Lagrange multiplier
  - λ is the marginal utility of an extra dollar of consumption expenditure
    - The marginal utility of income

$$\lambda = \frac{\partial U / \partial x_1}{p_1} = \frac{\partial U / \partial x_2}{p_2} = \ldots = \frac{\partial U / \partial x_n}{p_n}$$

The n-Good Case

- At the margin, the price of a good
  - Represents the consumer's evaluation of the utility of the last unit consumed
  - How much the consumer is willing to pay for the last unit

$$p_i = \frac{\partial U / \partial x_i}{\lambda}, \text{ for every } i$$

The n-Good Case

- Corner solutions
  - Means that the first-order conditions must be modified:

$$\partial \mathscr{L}/\partial x_i = \partial U/\partial x_i - \lambda p_i \leq 0 \quad (i = 1,\ldots,n)$$

  - If $\partial \mathscr{L}/\partial x_i = \partial U/\partial x_i - \lambda p_i < 0$, then $x_i = 0$
  - This means that

$$p_i > \frac{\partial U / \partial x_i}{\lambda} = \frac{MU_{x_i}}{\lambda}$$

    - any good whose price exceeds its marginal value to the consumer will not be purchased

EXAMPLE 4.1 Cobb–Douglas Demand Functions

- Cobb-Douglas utility function:

$$U(x,y) = x^{\alpha}y^{\beta}$$

- Setting up the Lagrangian:

$$\mathscr{L} = x^{\alpha}y^{\beta} + \lambda(I - p_x x - p_y y)$$

- First-order conditions:

$$\partial\mathscr{L}/\partial x = \alpha x^{\alpha-1}y^{\beta} - \lambda p_x = 0$$

$$\partial\mathscr{L}/\partial y = \beta x^{\alpha}y^{\beta-1} - \lambda p_y = 0$$

$$\partial\mathscr{L}/\partial \lambda = I - p_x x - p_y y = 0$$

EXAMPLE 4.1 Cobb–Douglas Demand Functions

- First-order conditions imply: $\alpha y/\beta x = p_x/p_y$
  - Since $\alpha + \beta = 1$: $p_y y = (\beta/\alpha)p_x x = [(1- \alpha)/\alpha]p_x x$
- Substituting into the budget constraint:

$$I = p_x x + [(1- \alpha)/\alpha]p_x x = (1/\alpha)p_x x$$

- Solving: $x^*=\alpha I/p_x$ and $y^*=\beta I/p_y$
  - The individual will allocate α percent of his income to good *x* and β percent of his income to good *y*

EXAMPLE 4.1 Cobb–Douglas Demand Functions

- Cobb-Douglas utility function
  - Is limited in its ability to explain actual consumption behavior
  - The share of income devoted to a good often changes in response to changing economic conditions
- A more general functional form might be more useful

EXAMPLE 4.2 CES Demand

- Assume that $\delta = 0.5$

$$U(x,y) = x^{0.5} + y^{0.5}$$

- Setting up the Lagrangian:

$$\mathscr{L} = x^{0.5} + y^{0.5} + \lambda(I - p_x x - p_y y)$$

- First-order conditions for a maximum:

$$\partial\mathscr{L}/\partial x = 0.5x^{-0.5} - \lambda p_x = 0$$

$$\partial\mathscr{L}/\partial y = 0.5y^{-0.5} - \lambda p_y = 0$$

$$\partial\mathscr{L}/\partial \lambda = I - p_x x - p_y y = 0$$

- This means that: $(y/x)^{0.5} = p_x/p_y$
  - Substituting into the budget constraint, we can solve for the demand functions

$$x^* = \frac{I}{p_x[1+(p_x/p_y)]} \qquad y^* = \frac{I}{p_y[1+(p_y/p_x)]}$$

- The share of income spent on either *x* or *y* is not a constant
  - Depends on the ratio of the two prices
  - The higher is the relative price of *x*, the smaller will be the share of income spent on *x*

EXAMPLE 4.2 CES Demand

- If δ = -1,

$$U(x,y) = -x^{-1} - y^{-1}$$

- First-order conditions imply that

$$y/x = (p_x/p_y)^{0.5}$$

- The demand functions are

$$x^* = \frac{I}{p_x\left[1+\left(\frac{p_y}{p_x}\right)^{0.5}\right]} \qquad y^* = \frac{I}{p_y\left[1+\left(\frac{p_x}{p_y}\right)^{0.5}\right]}$$

- If $\delta = -\infty$, $U(x,y) = Min(x,4y)$
  - The person will choose only combinations for which $x = 4y$
  - This means that

$$I = p_x x + p_y y = p_x x + p_y (x/4)$$

$$I = (p_x + 0.25 p_y) x$$

- The demand functions are

$$x^* = \frac{I}{p_x + 0.25 p_y} \qquad y^* = \frac{I}{4 p_x + p_y}$$

Indirect Utility Function

- It is often possible to manipulate first-order conditions to solve for optimal values of $x_1, x_2, \ldots, x_n$
  - These optimal values will be

$$x^*_1 = x_1(p_1, p_2, \ldots, p_n, I)$$
$$x^*_2 = x_2(p_1, p_2, \ldots, p_n, I)$$
$$\ldots$$
$$x^*_n = x_n(p_1, p_2, \ldots, p_n, I)$$

Indirect Utility Function

- We can use the optimal values of the *x*'s to find the indirect utility function

$$\text{maximum utility} = U[x^*_1(p_1,p_2,\ldots,p_n,I), x^*_2(p_1,p_2,\ldots,p_n,I),\ldots,x^*_n(p_1,p_2,\ldots,p_n,I)] =$$

$$= V(p_1,p_2,\ldots,p_n,I)$$

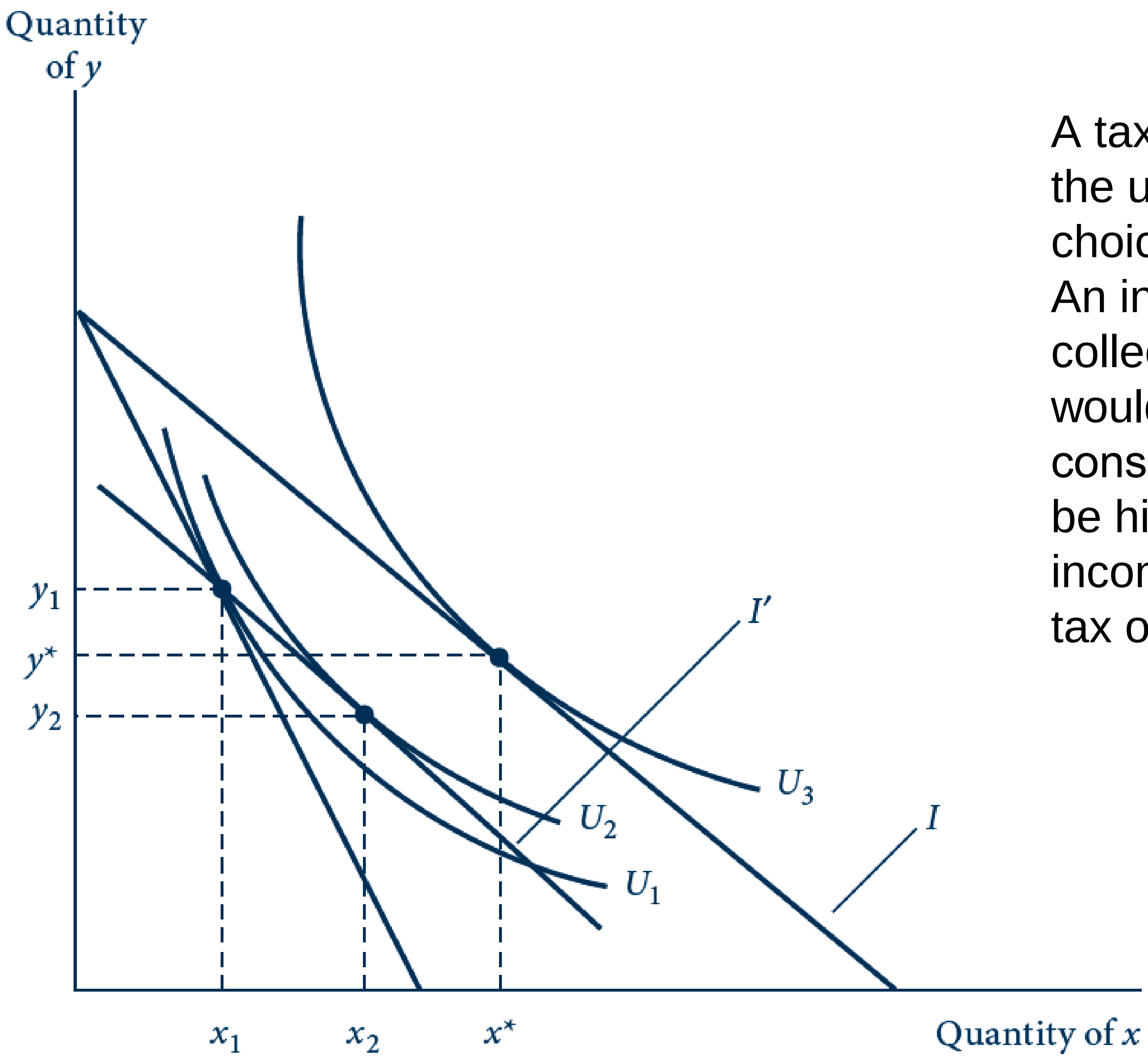
- The optimal level of utility will depend <u>indirectly</u> on prices and income

The Lump Sum Principle

- Taxes on an individual’s general purchasing power
  - Are superior to taxes on a specific good
    - An income tax allows the individual to decide freely how to allocate remaining income
    - A tax on a specific good will reduce an individual’s purchasing power and distort his choices

FIGURE 4.5

The Lump Sum Principle of Taxation



A tax on good x would shift the utility-maximizing choice from x*, y* to x_1, y_1. An income tax that collected the same amount would shift the budget constraint to I'. Utility would be higher (U_2) with the income tax than with the tax on x alone (U_1).

- Cobb-Douglas utility function
  - With $\alpha = \beta = 0.5$,
  - We know that

$$x^*=I/2p_x \text{ and } y^*=I/2p_y$$

- The indirect utility function

$$V(p_x,p_y,I) = (x^*)^{0.5}(y^*)^{0.5} = \frac{I}{2p_x^{0.5}p_y^{0.5}}$$

EXAMPLE 4.3 Indirect Utility and the Lump Sum Principle

- Fixed proportions

  $x^*=I/[p_x + 0.25p_y]$ and $y^*=I/[4p_x+p_y]$

- The indirect utility function

$$V(p_x, p_y, I) = \min(x^*, 4y^*) =$$

$$= x^* = \frac{I}{p_x + 0.25 p_y} =$$

$$= 4y^* = \frac{4}{4p_x + p_y}$$

EXAMPLE 4.3 Indirect Utility and the Lump Sum Principle

- The lump sum principle
- Cobb-Douglas
  - If a tax of $1 was imposed on good *x*
    - The individual will purchase $x^* = 2$
    - Indirect utility will fall from 2 to 1.41
  - An equal-revenue tax will reduce income to $6
    - Indirect utility will fall from 2 to 1.5

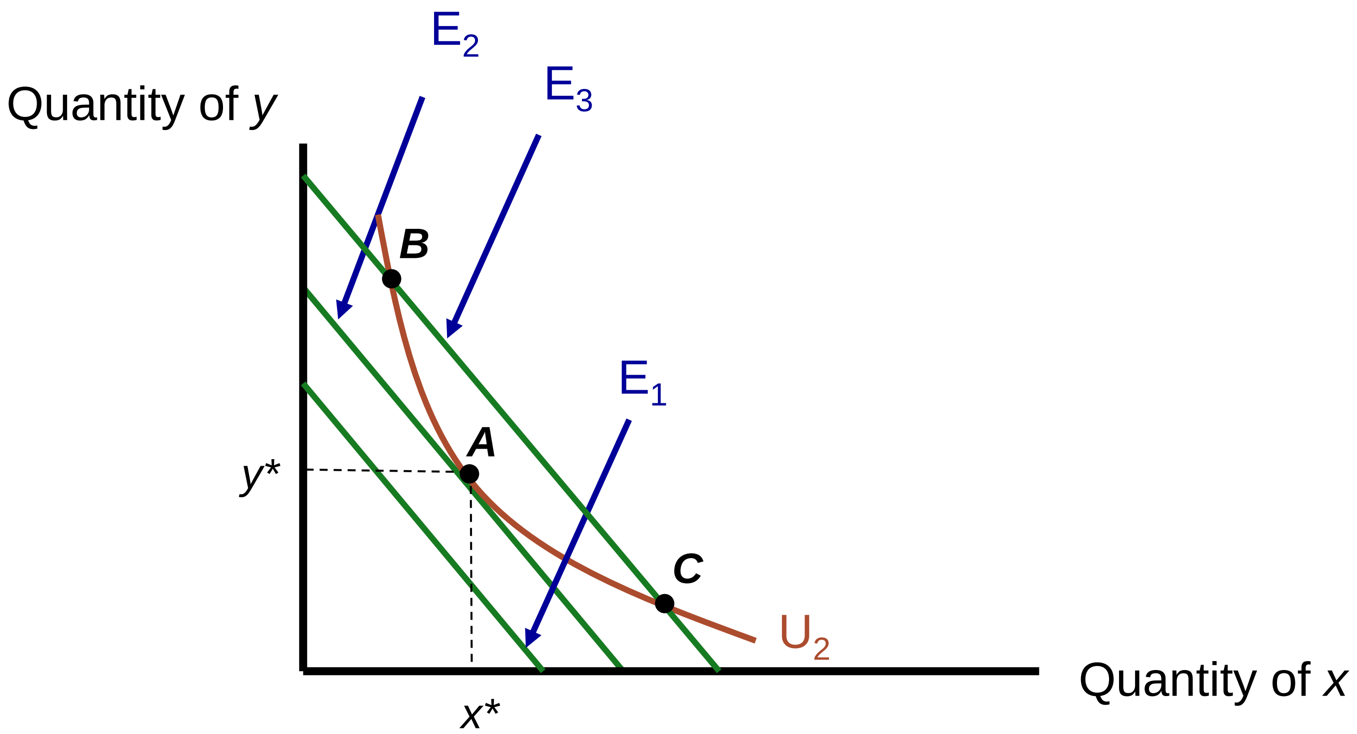
EXAMPLE 4.3 Indirect Utility and the Lump Sum Principle

- The lump sum principle
- Fixed-proportions
  - If a tax of $1 was imposed on good *x*
    - Indirect utility will fall from 4 to 8/3
  - An equal-revenue tax will reduce income to $16/3
    - Indirect utility will fall from 4 to 8/3
- Since preferences are rigid, the tax on *x* does not distort choices

Expenditure Minimization

- Dual minimization problem for utility maximization
  - Allocate income to achieve a given level of utility with the minimal expenditure
    - The goal and the constraint have been reversed

The Dual Expenditure-Minimization Problem



The dual of the utility-maximization problem is to attain a given utility level (U2) with minimal expenditures. An expenditure level of E1 does not permit U2 to be reached, whereas E3 provides more spending power than is strictly necessary. With expenditure E2, this person can just reach U2 by consuming x and y.

Expenditure Minimization

- The individual's problem is to choose $x_1, x_2, \ldots, x_n$ to minimize

$$\text{total expenditures} = E = p_1x_1 + p_2x_2 + \ldots + p_nx_n$$

subject to the constraint

$$\text{utility} = \bar{U} = U(x_1, x_2, \ldots, x_n)$$

  – The optimal amounts of $x_1, x_2, \ldots, x_n$ will depend on the prices of the goods and the required utility level

Expenditure Minimization

- Expenditure function
  - The individual's expenditure function
  - Shows the minimal expenditures
  - Necessary to achieve a given utility level
  - For a particular set of prices
- minimal expenditures = $E(p_1,p_2,\ldots,p_n,U)$

Expenditure Minimization

- The expenditure function and the indirect utility function
  - Are inversely related
  - Both depend on market prices
  - But involve different constraints

- Cobb-Douglas
  - The indirect utility function in the two-good case:

$$V(p_x, p_y, I) = \frac{I}{2 p_x^{0.5} p_y^{0.5}}$$

  - If we interchange the role of utility and income (expenditure), we will have the expenditure function

$$E(p_x, p_y, U) = 2 p_x^{0.5} p_y^{0.5} U$$

- Fixed-proportions case
  - The indirect utility function:

$$V(p_x, p_y, I) = \frac{I}{p_x + 0.25 p_y}$$

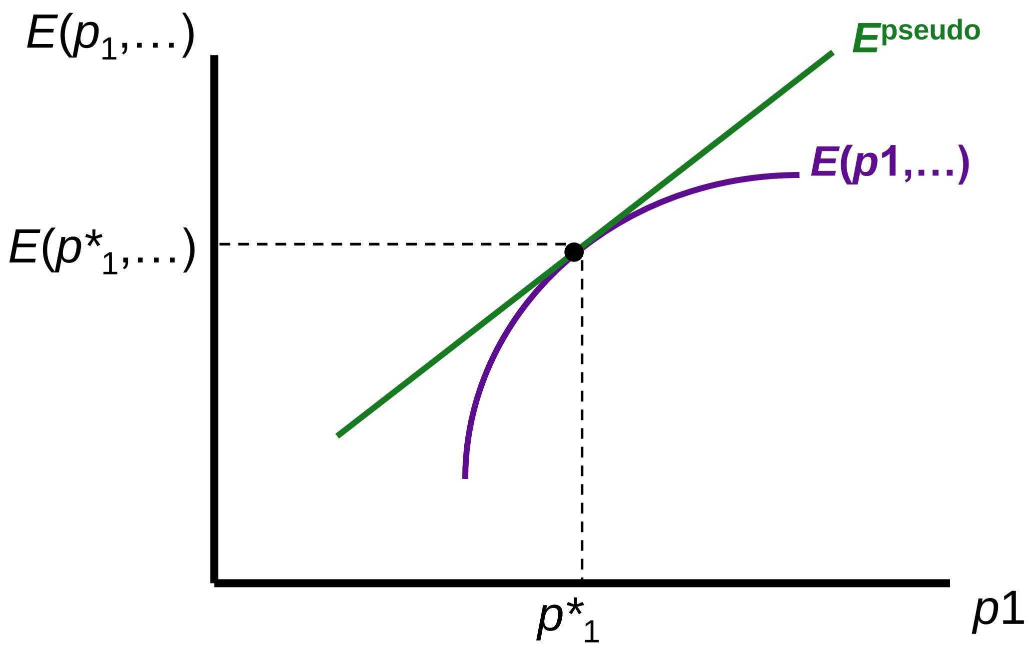
    - If we interchange the role of utility and income (expenditure), we will have the expenditure function

$$E(p_x, p_y, U) = (p_x + 0.25 p_y)U$$

Properties of Expenditure Functions

- Homogeneity
  - A doubling of all prices will precisely double the value of required expenditures
    - Homogeneous of degree one
- Nondecreasing in prices
  - $\partial E/\partial p_i \geq 0$ for every good, *i*
- Concave in prices
  - Functions that always lie below tangents to them

Expenditure Functions Are Concave in Prices



At p_1 this person spends $E(p_1^*, \ldots)$. If he or she continues to buy the same set of goods as p_1 changes, then expenditures would be given by E^{pseudo}. Because his or her consumption patterns will likely change as p_1 changes, actual expenditures will be less than this.

Budget Shares

- Engel's law
  - Fraction of income spent on food decreases as income increases
- Budget shares, $s_i = p_i x_i / I$
- Recent budget share data
  - Engel's law is clearly visible
- Cobb–Douglas utility function
  - Is not useful for detailed empirical studies of household behavior

TABLE E4.1

Budget shares of U.S. households, 2008

	Annual Income		
	$10,000–$14,999	$40,000–$49,999	Over $70,000
Expenditure Item			
Food	15.7	13.4	11.8
Shelter	23.1	21.2	19.3
Utilities, fuel, and public services	11.2	8.6	5.8
Transportation	14.1	17.8	16.8
Health insurance	5.3	4.0	2.6
Other health-care expenses	2.6	2.8	2.3
Entertainment (including alcohol)	4.6	5.2	5.8
Education	2.3	1.2	2.6
Insurance and pensions	2.2	8.5	14.6
Other (apparel, personal care, other housing expenses, and misc.)	18.9	17.3	18.4

EXTENSIONS

Linear expenditure system

- Generalization of the Cobb–Douglas function
  - Incorporates the idea that certain minimal amounts of each good must be bought by an individual (x_0, y_0)

$$U(x,y)=(x-x_0)^{\alpha}(y-y_0)^{\beta}$$

    - For $x \geq x_0$ and $y \geq y_0$,
    - Where $\alpha+\beta=1$

Linear expenditure system

- Supernumerary income (I*)
  - Amount of purchasing power remaining after purchasing the minimum bundle

$I^{*}=I-p_{x}x_{0}-p_{y}y_{0}$

- The demand functions are:

$x = (p_{x}x_{0}+\alpha I^{*})/p_{x}$ and $y = (p_{y}y_{0}+\beta I^{*})/p_{y}$

  - The share equations:

    $s_{x}= \alpha+(\beta p_{x}x_{0}-\alpha p_{y}y_{0})/I$

    $s_{x}= \beta+(\alpha p_{y}y_{0} -\beta p_{x}x_{0})/I$

    - Not homothetic

CES utility

- CES utility function

$$U(x,y)=\frac{x^\delta}{\delta}+\frac{y^\delta}{\delta},\ \text{for } \delta\le 1, \delta\ne 0$$

- Budget shares:

  $s_x=1/[1+(p_y/p_x)^K]$ and $s_y=1/[1+(p_x/p_y)^K]$

  - Where $K = \delta/(\delta-1)$

- Homothetic

The almost ideal demand system

- Expenditure functions
  - Logarithmic differentiation

$$\frac{\partial \ln E(p_x, p_y, V)}{\partial \ln p_x} = \frac{1}{E(p_x, p_y, V)} \cdot \frac{\partial E}{\partial p_x} \cdot \frac{\partial p_x}{\partial \ln p_x} = \frac{xp_x}{E} = s_x$$

- AIDS: first-order approximation to any demand system
- Reference:
  - Angus Deaton and John Muellbauer (1980), “An Almost Ideal Demand System,” *The American Economic Review*, Vol. 70, No. 3. (Jun., 1980), pp. 312-326.

The almost ideal demand system

- Almost ideal demand system
  - Expenditure function

$$\ln E(p_x, p_y, V) = a_0 + a_1 \ln p_x + a_2 \ln p_y +$$
$$+0.5b_1(\ln p_x)^2 + b_2 \ln p_x \ln p_y +$$
$$+0.5b_3(\ln p_y)^2 + Vc_0 p_x^{c_1} p_y^{c_2}$$

- Almost ideal demand system
  - Expenditure function
- Homogeneous of degree one in the prices
  - $a_1+a_2=1$, $b_1+b_2=0$, and $c_1+c_2=0$

The almost ideal demand system

$$s_x = a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1 V c_0 p_x^{c_1} p_y^{c_2}$$

$$s_y = a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2 V c_0 p_x^{c_1} p_y^{c_2}$$

$$s_x + s_y = 1$$

$$s_x = a_1 + b_1 \ln p_x + b_2 \ln p_y + c_1 (E / p)$$

$$s_y = a_2 + b_2 \ln p_x + b_3 \ln p_y + c_2 (E / p)$$

p is an index of prices

$$\ln p = a_0 + a_1 \ln p_x + a_2 \ln p_y + 0.5 b_1 (\ln p_x)^2 +$$

$$+ b_2 \ln p_x \ln p_y + 0.5 b_3 (\ln p_y)^2$$

What are we going to do today?

- Under two-good world:
  - Income/substitution Effect (Slutsky's equation)
  - Hicksian Demand
  - Elasticity
  - Consumer Surplus
  - Revealed Demand
- Generalization to n-good

Income and Substitution Effect

Reading

- Reading
  - Chapter 5

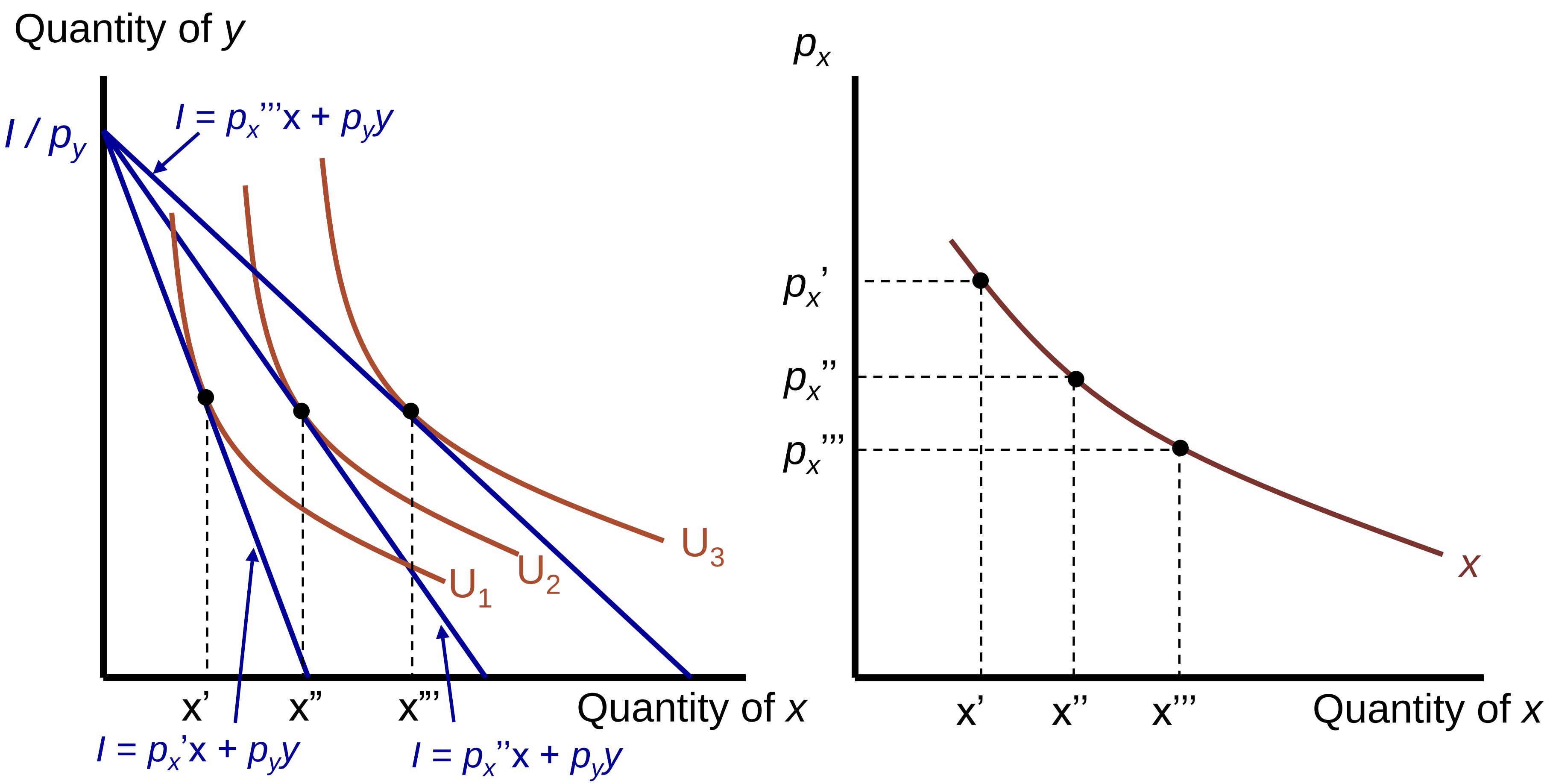
Recall

- Utility Maximization Problem
  - Optimal solution: Marshallian Demand
  - Optimal-value function: Indirect utility function
- Expenditure Minimization Problem
  - Optimal Solution: Hicksian Demand
  - Optimal-value function : Expenditure function

Marshallian Demand function

- Depend on price and income
- Draw demand curve (Econ 101)
- Comparative Statics:
  - Change in price?
  - Change in income?

Construction of an Individual's Demand Curve



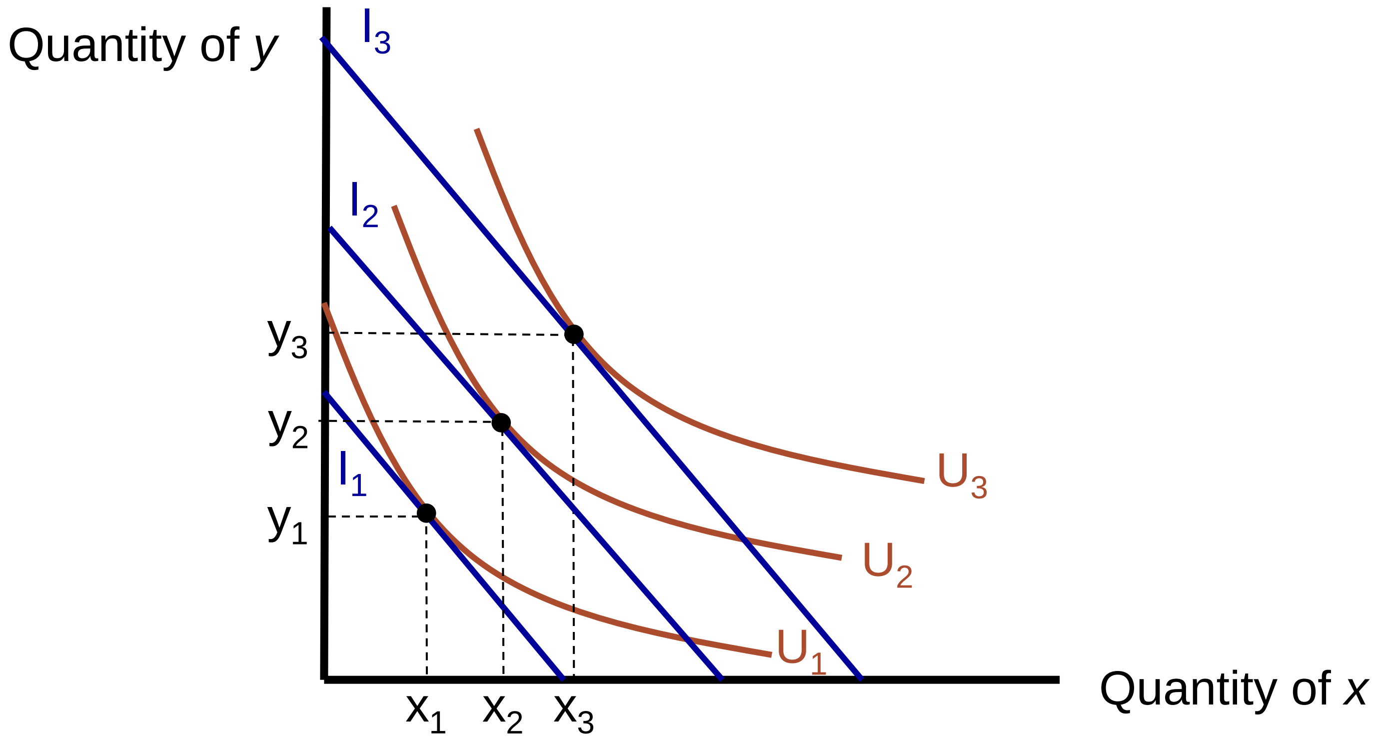
In (a), the individual's utility-maximizing choices of x and y are shown for three different prices of x(p'_x, p''_x, and p'''_x). In (b), this relationship between p_x and x is used to construct the demand curve for x. The demand curve is drawn on the assumption that p_y, I, and preferences remain constant as p_x varies.

Change in Income

- Normal good: $\partial x_i/\partial I \geq 0$
- Inferior good: $\partial x_i/\partial I < 0$

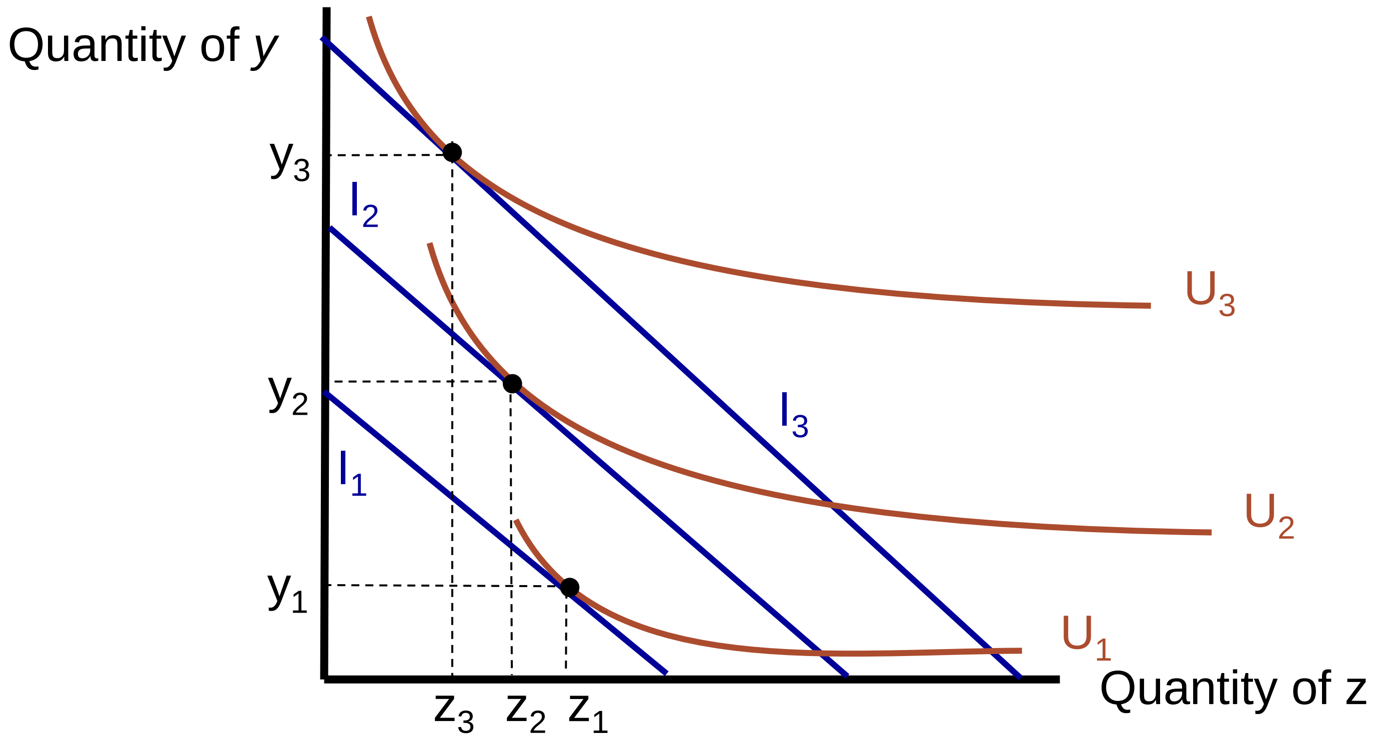
FIGURE 5.1

Effect of an Increase in Income on the Quantities of x and y Chosen



As income increases from I_1 to I_2 to I_3, the optimal (utility-maximizing) choices of x and y are shown by the successively higher points of tangency. Observe that the budget constraint shifts in a parallel way because its slope (given by $-p_x/p_y$) does not change.

An Indifference Curve Map Exhibiting Inferiority

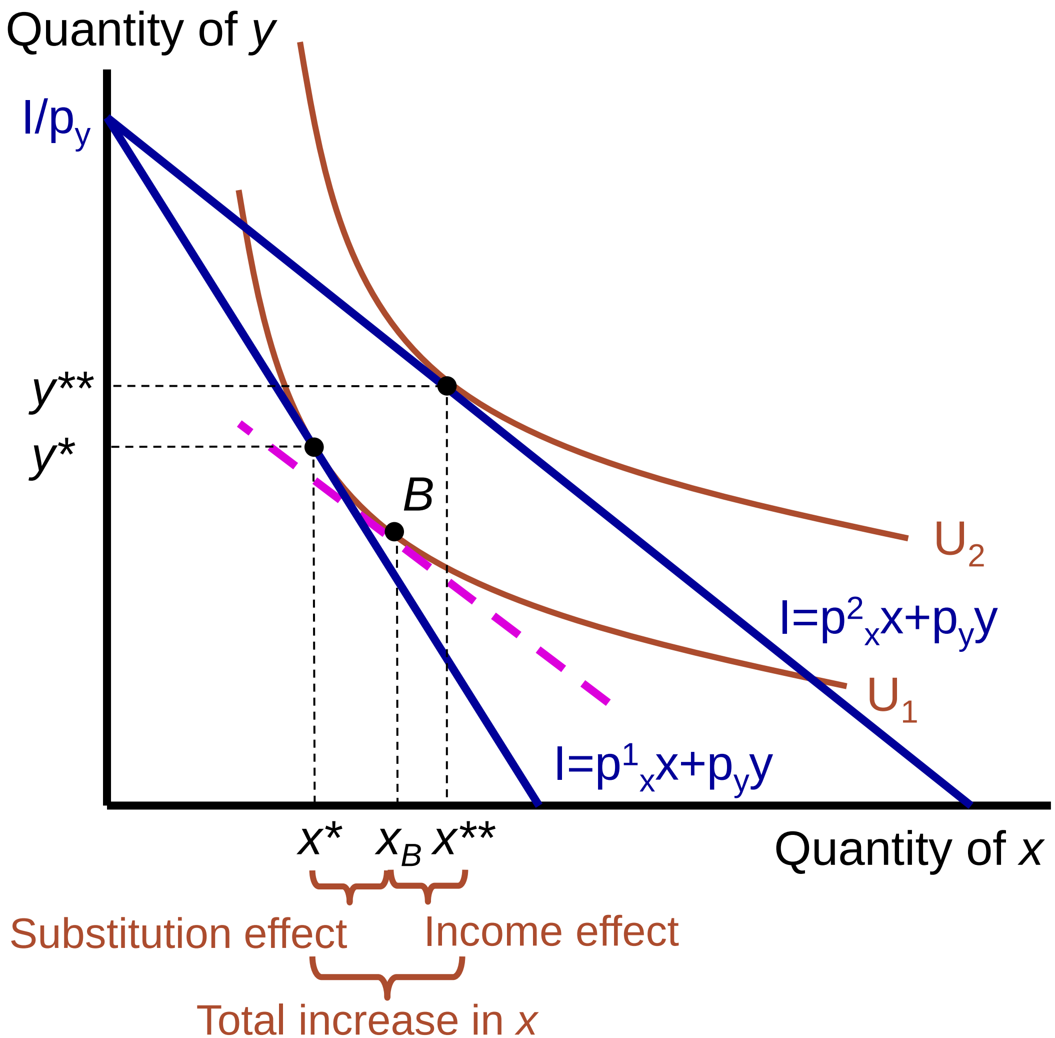


In this diagram, good z is inferior because the quantity purchased decreases as income increases. Here, y is a normal good (as it must be if there are only two goods available), and purchases of y increase as total expenditures increase.

Change in Price

- Two effects
  - Substitution effect (change in relative price)
  - Income effect (real purchasing power is different)
- Price increases
  - Substitution: always negative
  - Income: depends
- Possibility of Giffen goods (P↑ & Q↑)

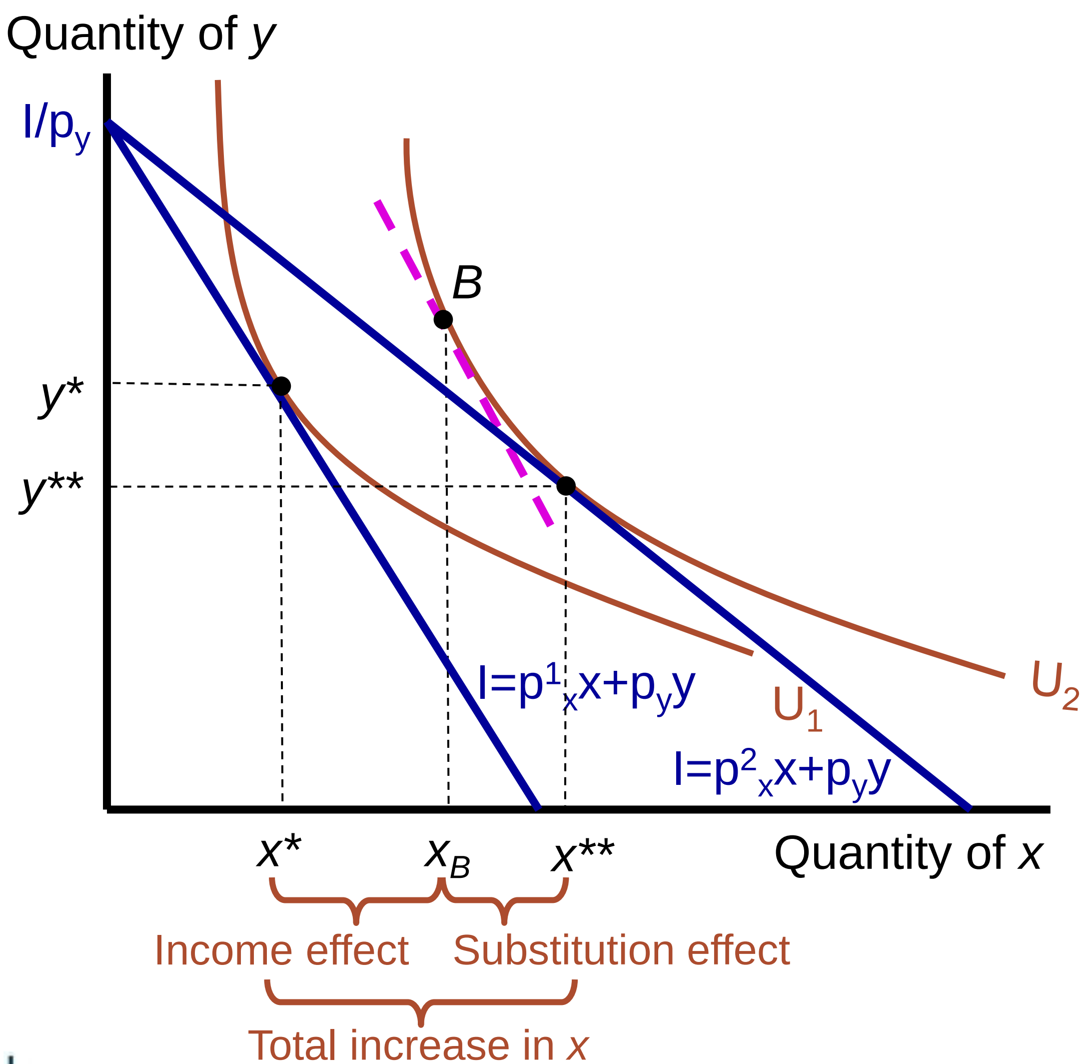
Demonstration of the Income and Substitution Effects of a Decrease in the Price of x



When the price of x decreases from p^1_x to p^2_x , the utility-maximizing choice shifts from x*,y* to x**,y**. This movement can be broken down into two analytically different effects: first, the substitution effect, involving a movement along the initial indifference curve to point B, where the MRS is equal to the new price ratio; and second, the income effect, entailing a movement to a higher level of utility because real income has increased. In the diagram, both the substitution and income effects cause more x to be bought when its price decreases. Notice that point I/p_y is the same as before the price change; this is because p_y has not changed. Therefore, point I/p_y appears on both the old and new budget constraints.

FIGURE 5.4

Demonstration of the Income and Substitution Effects of an Increase in the Price of x



When the price of x increases, the budget constraint shifts inward. The movement from the initial utility-maximizing point (x**,y**) to the new point (x*,y*) can be analyzed as two separate effects. The substitution effect would be depicted as a movement to point B on the initial indifference curve (U_2). The price increase, however, would create a loss of purchasing power and a consequent movement to a lower indifference curve. This is the income effect. In the diagram, both the income and substitution effects cause the quantity of x to decrease as a result of the increase in its price. Again, the point I/p_y is not affected by the change in the price of x.

Classwork

- For positive income effect
  - Graph for price increase
  - Graph for price decrease
- For negative income effect
  - Graph for price increase
  - Graph for price decrease
- Draw graph for Giffen paradox

Change together?

- Homogenous of degree zero in prices and income
  - double price and income results in no change
  - No money illusion
  - $x_i^* = x_i(p_1,p_2,\ldots,p_n,I) = x_i(tp_1,tp_2,\ldots,tp_n,tI)$ for $t>0$

EXAMPLE 5.1 Homogeneity

- Cobb-Douglas utility function:

  utility = $U(x,y) = x^{0.3}y^{0.7}$

  - The demand functions are: $x^* = 0.3I/p_x$ and

  $y^* = 0.7I/p_y$

  - Homogeneity

- CES utility function:

  utility = $U(x,y) = x^{0.5} + y^{0.5}$

  - The demand functions are:

$$x^* = \frac{1}{1 + p_x/p_y} \cdot \frac{I}{p_x} \qquad y^* = \frac{1}{1 + p_y/p_x} \cdot \frac{I}{p_y}$$

  - Homogeneity

Hicksian Demand

- Demand of a good when utility is held constant
  - instead of budget is constant in Marshallian demand
  - Assuming that other prices and utility are held constant
- Reflects substitution effect only
- For two goods x,y,

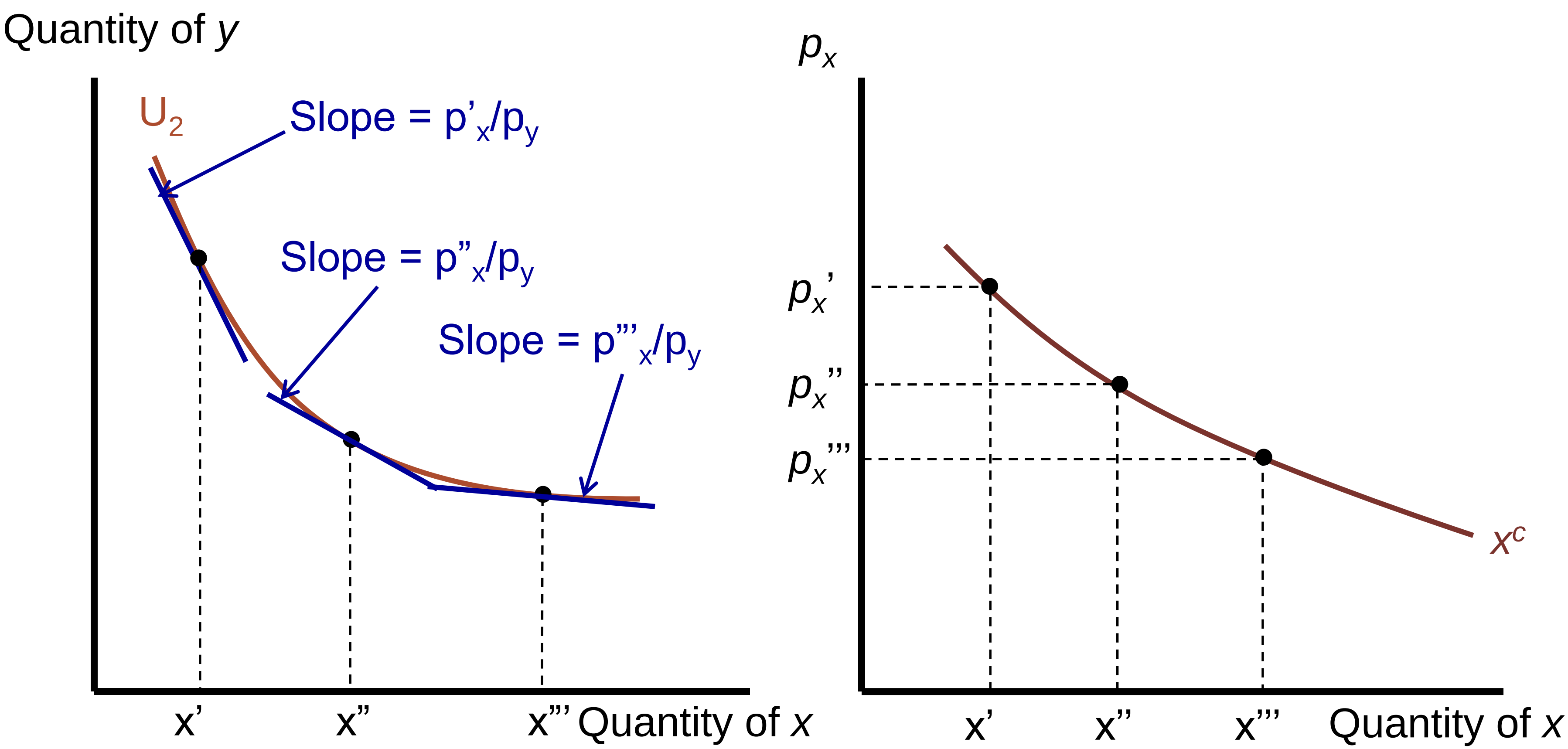
$$x^* = x^c(p_x, p_y, U)$$

Finding Hicksian demand

- Direct:
  - Solution to Expenditure minimization problem
- Indirect: (Shephard's Lemma)
  - Envelop theorem: differentiate the expenditure function with respect to good's price

$$\frac{\partial E(p_x, p_y, U)}{\partial p_x} = \frac{\partial L}{\partial p_x} = x^c(p_x, p_y, U)$$

Construction of a Compensated Demand Curve



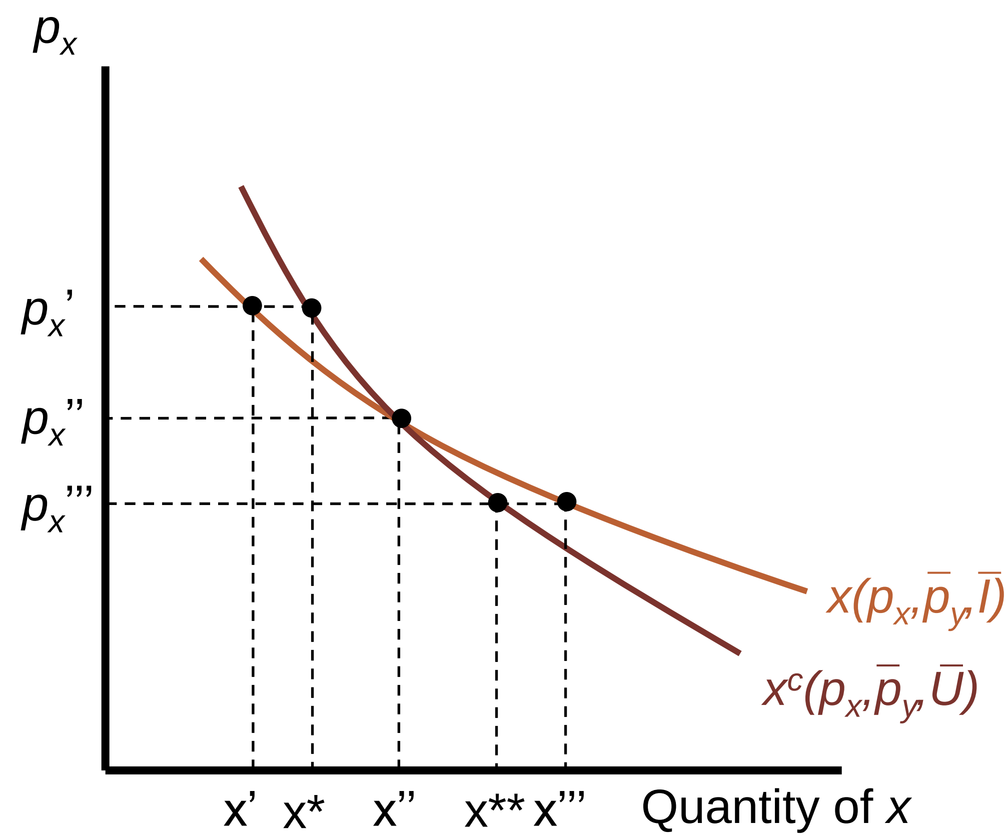
The curve x^c shows how the quantity of x demanded changes when p_x changes, holding p_y and utility constant. That is, the individual's income is "compensated" to keep utility constant. Hence x^c reflects only substitution effects of changing prices.

Compensated (HICKSIAN) Demand Curves and Functions

- Relationship between compensated and uncompensated demand curves
  - Normal good
    - Compensated demand curve is less responsive to price changes than is the uncompensated demand curve
      - Uncompensated demand curve reflects both income and substitution effects
      - Compensated demand curve reflects only substitution effects

FIGURE 5.7

Comparison of Compensated and Uncompensated Demand Curves



The compensated (x^c) and uncompensated (x) demand curves intersect at p''_x because x'' is demanded under each concept. For prices above p''_x, the individual's purchasing power must be increased with the compensated demand curve; thus, more x is demanded than with the uncompensated curve. For prices below p''_x, purchasing power must be reduced for the compensated curve; therefore, less x is demanded than with the uncompensated curve. The standard demand curve is more price-responsive because it incorporates both substitution and income effects, whereas the curve x^c reflects only substitution effects.

Compensated Demand Functions

- The utility is: utility = $U(x,y) = x^{0.5}y^{0.5}$
  - The Marshallian demand functions:

$x(p_x,p_y,I) = 0.5I/p_x$ and $y(p_x,p_y,I) = 0.5I/p_y$

  - The expenditure function: $E(p_x,p_y, U)=2p_x^{0.5}p_y^{0.5}U$
  - Use Shephard's lemma to calculate the compensated demand functions as

$$x^c(p_x, p_y, U) = \frac{\partial E(p_x, p_y, U)}{\partial p_x} = p_x^{-0.5} p_y^{0.5} U$$

$$y^c(p_x, p_y, U) = \frac{\partial E(p_x, p_y, U)}{\partial p_y} = p_x^{0.5} p_y^{-0.5} U$$

Slutsky's Equation

- How marshallian and Hicksian are related?
  - $x^c(p_x,p_y,U) = x[p_x,p_y,E(p_x,p_y,U)]$
- Differentiate both sides by p_x

$$\frac{\partial x^c}{\partial p_x} = \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial E}\cdot\frac{\partial E}{\partial p_x},\ \text{or:}\ \frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E}\cdot\frac{\partial E}{\partial p_x}$$

A Mathematical Development of Response to Price Changes

- The Slutsky equation

$$\text{substitution effect} = \frac{\partial x^c}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}}$$

$$\text{income effect} = -\frac{\partial x}{\partial E}\cdot\frac{\partial E}{\partial p_x} = -\frac{\partial x}{\partial I}\cdot\frac{\partial E}{\partial p_x} = -x^c\,\frac{\partial x}{\partial I}$$

Slutsky equation: substitution effect + income effect

$$\frac{\partial x(p_x, p_y, I)}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}} - x^c\,\frac{\partial x}{\partial I}$$

A Mathematical Development of Response to Price Changes

- The Slutsky equation
  - The substitution effect
    - Always negative as long as *MRS* is diminishing
    - The slope of the compensated demand curve must be negative
  - The income effect
    - If x is a normal good, then $\partial x/\partial I > 0$
      - The entire income effect is negative
    - If x is an inferior good, then $\partial x/\partial I < 0$
      - The entire income effect is positive

EXAMPLE 5.4 A Slutsky Decomposition

- Marshallian demand function for good x

$$x(p_x,p_y,I) = 0.5I/p_x$$

- Compensated demand function for this good

$$x^c(p_x,p_y, U)=p_x^{-0.5}p_y^{0.5}U$$

- Total effect of a price change on Marshallian demand

$$\frac{\partial x(p_x, p_y, I)}{\partial p_x} = \frac{-0.5I}{p_x^2}$$

$$\text{substitution effect} = \frac{\partial x^c(p_x, p_y, U)}{\partial p_x} = -0.5 p_x^{-1.5} p_y U =$$

$$= -0.5 p_x^{-1.5} p_y V = -0.25 p_x^{-2} I$$

indirect utility function $V(p_x, p_y, I) = 0.5 I p_x^{-0.5} p_y^{-0.5}$

$$\text{income effect} = -x \frac{\partial x}{\partial I} = -\left[\frac{0.5I}{p_x}\right] \cdot \frac{0.5}{p_x} = -\frac{0.25I}{p_x^2}$$

Marshallian Demand Elasticities

- Marshallian demand elasticities
  - Marshallian demand function: $x(p_x, p_y, I)$

1. Price elasticity of demand, e_{x, p_x}
   - Measures the proportionate change in quantity demanded
     - In response to a proportionate change in a good's own price

$$e_{x,p_x} = \frac{\Delta x / x}{\Delta p_x / p_x} = \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{x}$$

Marshallian Demand Elasticities

2. Income elasticity of demand, $e_{x,I}$

- Measures the proportionate change in quantity demanded
  - In response to a proportionate change in income

$$e_{x,I} = \frac{\Delta x / x}{\Delta I / I} = \frac{\partial x}{\partial I} \cdot \frac{I}{x}$$

Marshallian Demand Elasticities

3. Cross-price elasticity of demand, $e_{x,py}$

- Measures the proportionate change in quantity of x demanded
  - In response to a proportionate change in the price of some other good (y)

$$e_{x,p_y} = \frac{\Delta x / x}{\Delta p_y / p_y} = \frac{\partial x}{\partial p_y} \cdot \frac{p_y}{x}$$

Price elasticity of demand

- The own price elasticity of demand is always negative
  - The only exception is Giffen's paradox
- The size of the elasticity is important
  - If $e_{x,p_X} < -1$, demand is elastic
  - If $e_{x,p_X} > -1$, demand is inelastic
  - If $e_{x,p_X} = -1$, demand is unit elastic

Price Elasticity of Demand

- Total spending on $x = p_x x$

$$\frac{\partial(p_x x)}{\partial p_x} = p_x \cdot \frac{\partial x}{\partial p_x} + x = x[e_{x,p_x} + 1]$$

- If $e_{x,p_x} > -1$, demand is inelastic
  – Price and total spending move in the same direction
- If $e_{x,p_x} < -1$, demand is elastic
  – Price and total spending move in opposite directions

Compensated Price Elasticities

- Compensated price elasticities
  - Compensated demand function, $x^c(p_x,p_y,U)$

1. Compensated own-price elasticity of demand, $e_{xc,px}$
   - Measures the proportionate compensated change in quantity demanded
     - In response to a proportionate change in a good's own price

Compensated Price Elasticities

2. Compensated cross-price elasticity of demand, $e_{xc,py}$
   - Measures the proportionate compensated change in quantity of x demanded
     - In response to a proportionate change in the price of another good, y

Compensated Price Elasticities

$$e_{x^c,p_x} = \frac{\Delta x^c / x^c}{\Delta p_x / p_x} = \frac{\Delta x^c}{\Delta p_x} \cdot \frac{p_x}{x^c} = \frac{\partial x^c(p_x, p_y, U)}{\partial p_x} \cdot \frac{p_x}{x^c}$$

$$e_{x^c,p_y} = \frac{\Delta x^c / x^c}{\Delta p_y / p_y} = \frac{\Delta x^c}{\Delta p_y} \cdot \frac{p_y}{x^c} = \frac{\partial x^c(p_x, p_y, U)}{\partial p_y} \cdot \frac{p_y}{x^c}$$

using the Slutsky equation:

$$\frac{p_x}{x} \cdot \frac{\partial x}{\partial p_x} = e_{x,p_x} = \frac{p_x}{x^c} \cdot \frac{\partial x^c}{\partial p_x} - \frac{p_x}{x} \cdot x \cdot \frac{\partial x}{\partial I} = e^c_{x,p_x} - s_x e_{x,I}$$

where $\mathrm{s_x = p_x x / I}$ is the share of total income devoted to the purchase of good *x*

Compensated Price Elasticities

- The Slutsky equation shows
  - That the compensated and uncompensated price elasticities will be similar if
    - The share of income devoted to *x* is small
    - The income elasticity of *x* is small

Relationships among Demand Elasticities

- Homogeneity
  - Demand functions are homogeneous of degree zero in all prices and income
  - Euler's theorem for homogenous functions shows that

$$0 = p_x \cdot \frac{\partial x}{\partial p_x} + p_y \cdot \frac{\partial x}{\partial p_y} + I \cdot \frac{\partial x}{\partial I}$$

divide by x:

$$0 = e_{x,p_x} + e_{x,p_y} + e_{x,I}$$

Relationships among Demand Elasticities

- Engel aggregation
  - Engel's law: income elasticity of demand for food items is <1
    - Income elasticity of demand for all nonfood items must be >1
  - Differentiate the budget constraint with respect to income

$$1 = p_x \cdot \frac{\partial x}{\partial I} + p_y \cdot \frac{\partial y}{\partial I}$$

$$1 = p_x \cdot \frac{\partial x}{\partial I} \cdot \frac{xI}{xI} + p_y \cdot \frac{\partial y}{\partial I} \cdot \frac{yI}{yI} = s_x e_{x,I} + s_y e_{y,I}$$

Relationships among Demand Elasticities

- Cournot aggregation
  - The size of the cross-price effect of a change in p_x on the quantity of y consumed is restricted because of the budget constraint
  - Differentiate the budget constraint with respect to p_x

Relationships among Demand Elasticities

- Cournot aggregation

$$\frac{\partial I}{\partial p_x} = 0 = p_x \cdot \frac{\partial x}{\partial p_x} + x + p_y \cdot \frac{\partial y}{\partial p_x}$$

$$0 = p_x \cdot \frac{\partial x}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{x}{x} + x \cdot \frac{p_x}{I} + p_y \cdot \frac{\partial y}{\partial p_x} \cdot \frac{p_x}{I} \cdot \frac{y}{y}$$

$$0 = s_x e_{x,p_x} + s_x + s_y e_{y,p_x}$$

$$s_x e_{x,p_x} + s_y e_{y,p_x} = -s_x$$

EXAMPLE 5.5 Demand Elasticities: The Importance of Substitution Effects

1. Cobb-Douglas: $U(x,y)=x^{\alpha}y^{\beta}$, $\alpha+\beta=1$

- Demand functions: $x(p_x,p_y,I)=\alpha I/p_x$, and

  $y(p_x,p_y,I)=\beta I/p_y=(1-\alpha)I/p_y$

- Elasticities:

$$e_{x,p_x}=\frac{\partial x}{\partial p_x}\cdot\frac{p_x}{x}=-\frac{\alpha I}{p_x^2}\cdot\frac{p_x}{x}=-1$$

$$e_{x,p_y}=\frac{\partial x}{\partial p_y}\cdot\frac{p_y}{x}=0\cdot\frac{p_y}{x}=0$$

$$e_{x,I}=\frac{\partial x}{\partial I}\cdot\frac{I}{x}=\frac{\alpha}{p_x}\cdot\frac{I}{(\partial I/\partial p_x)}=1$$

EXAMPLE 5.5 Demand Elasticities: The Importance of Substitution Effects

1. Cobb-Douglas:

- Because $s_x = \alpha$ and $s_y = \beta$

Homogeneity:

$$e_{x,p_x} + e_{x,p_y} + e_{x,I} = -1 + 0 + 1 = 0$$

Engel aggregation :

$$s_x e_{x,I} + s_y e_{y,I} = \alpha \cdot 1 + \beta \cdot 1 = \alpha + \beta = 1$$

Cournot aggregation :

$$s_x e_{x,p_x} + s_y e_{y,p_x} = \alpha \cdot (-1) + \beta \cdot 0 = -\alpha = -s_x$$

EXAMPLE 5.5 Demand Elasticities: The Importance of Substitution Effects

1. Cobb-Douglas:

- Using the Slutsky equation in elasticity form to derive the compensated price elasticity:

$$e^c_{x,p_x} = e_{x,p_x} + s_x e_{x,I} = -1 + \alpha(1) = \alpha - 1 = -\beta$$

EXAMPLE 5.5 Demand Elasticities: The Importance of Substitution Effects

2. CES utility function (σ = 2, δ = 5): $U(x,y) = x^{0.5} + y^{0.5}$

- The demand functions for x and y:

$x(p_x,p_y,I)=I/p_x(1+ p_xp_y^{-1})$; $y(p_x,p_y,I) =I/p_x(1+ p_x^{-1}p_y)$

- We will use the "share elasticity" to derive the own price elasticity

$$e_{s_x,p_x} = \frac{\partial s_x}{\partial p_x} \cdot \frac{p_x}{s_x} = 1 + e_{x,p_x}$$

in this case: $s_x = \dfrac{p_x x}{I} = \dfrac{1}{1 + p_x p_y^{-1}}$

2. CES utility function (σ = 2, δ = 5)

- The share elasticity is given by

$$e_{s_x, p_x} = \frac{\partial s_x}{\partial p_x} \cdot \frac{p_x}{s_x} = \frac{-p_y^{-1}}{(1 + p_x p_y^{-1})^2} \cdot \frac{p_x}{(1 + p_x p_y^{-1})^{-1}} = \frac{-p_x p_y^{-1}}{1 + p_x p_y^{-1}}$$

3. CES utility function (σ = 0.5, δ = -1):

$U(x,y) = -x^{-1} - y^{-1}$

- The share of good *x* is: $s_x = \frac{p_x x}{I} = \frac{1}{1 + p_y^{0.5} p_x^{-0.5}}$

- The share elasticity is

$$e_{s_x, p_x} = \frac{\partial s_x}{\partial p_x} \cdot \frac{p_x}{s_x} = \frac{0.5 p_y^{0.5} p_x^{-1.5}}{(1 + p_y^{0.5} p_x^{-0.5})^2} \cdot \frac{p_x}{(1 + p_y^{0.5} p_x^{-0.5})^{-1}} = \frac{0.5 p_y^{0.5} p_x^{-0.5}}{1 + p_y^{0.5} p_x^{-0.5}}$$

- In general, the compensated price elasticity is

$$e_{x^c, p_x} = -(1 - s_x)\sigma$$

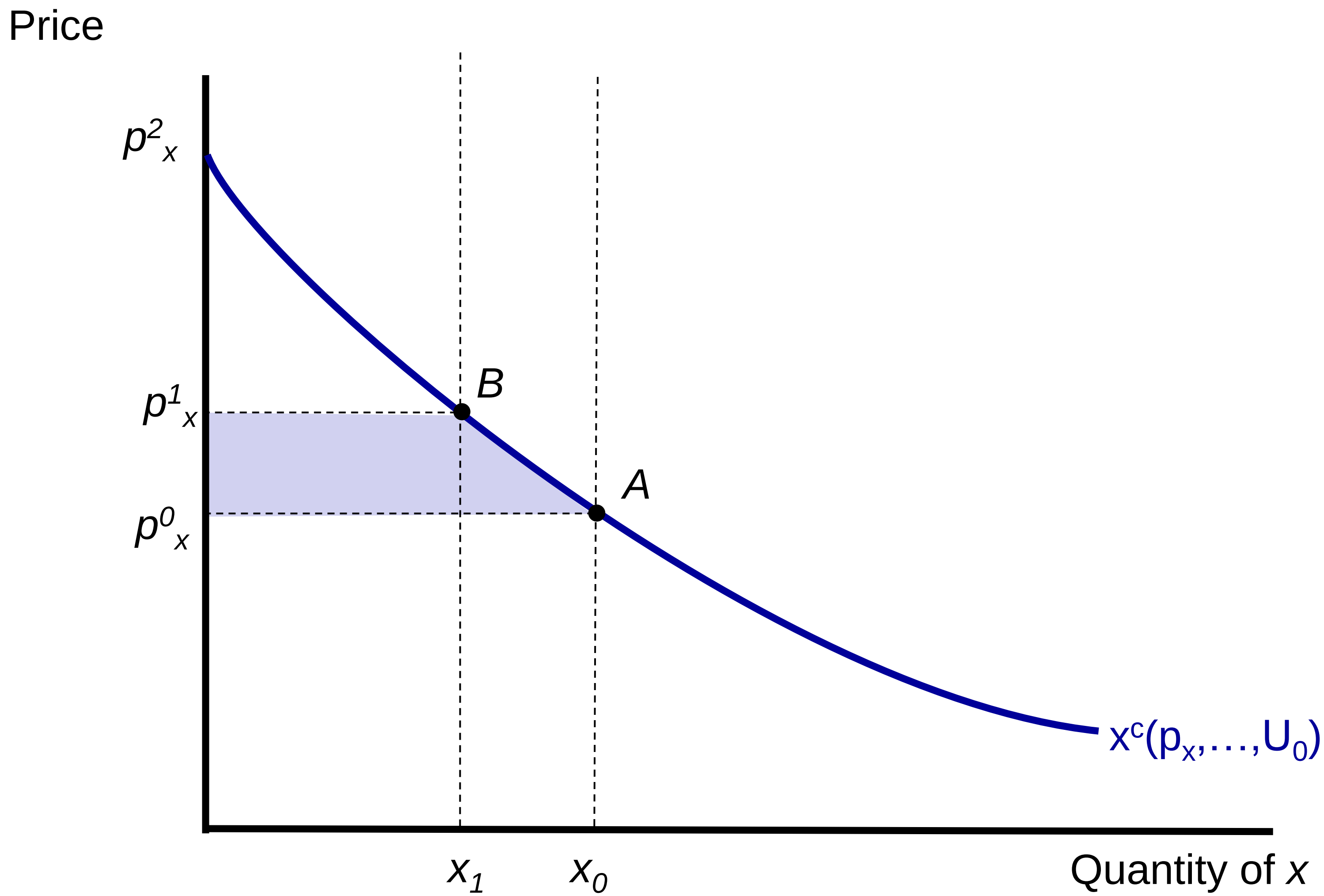
Welfare changes

- How to measure welfare changes if price of good x increases from p^0_x to p^1_x?

Consumer surplus

- Area under Marshallian demand curve

Showing Compensating Variation



If the price of x increases from p^0_x to p^1_x, this person needs extra expenditures of CV to remain on the U_0 indifference curve. Integration shows that CV can also be represented by the shaded area below the compensated demand curve in panel (b).

Compensating variation (CV)

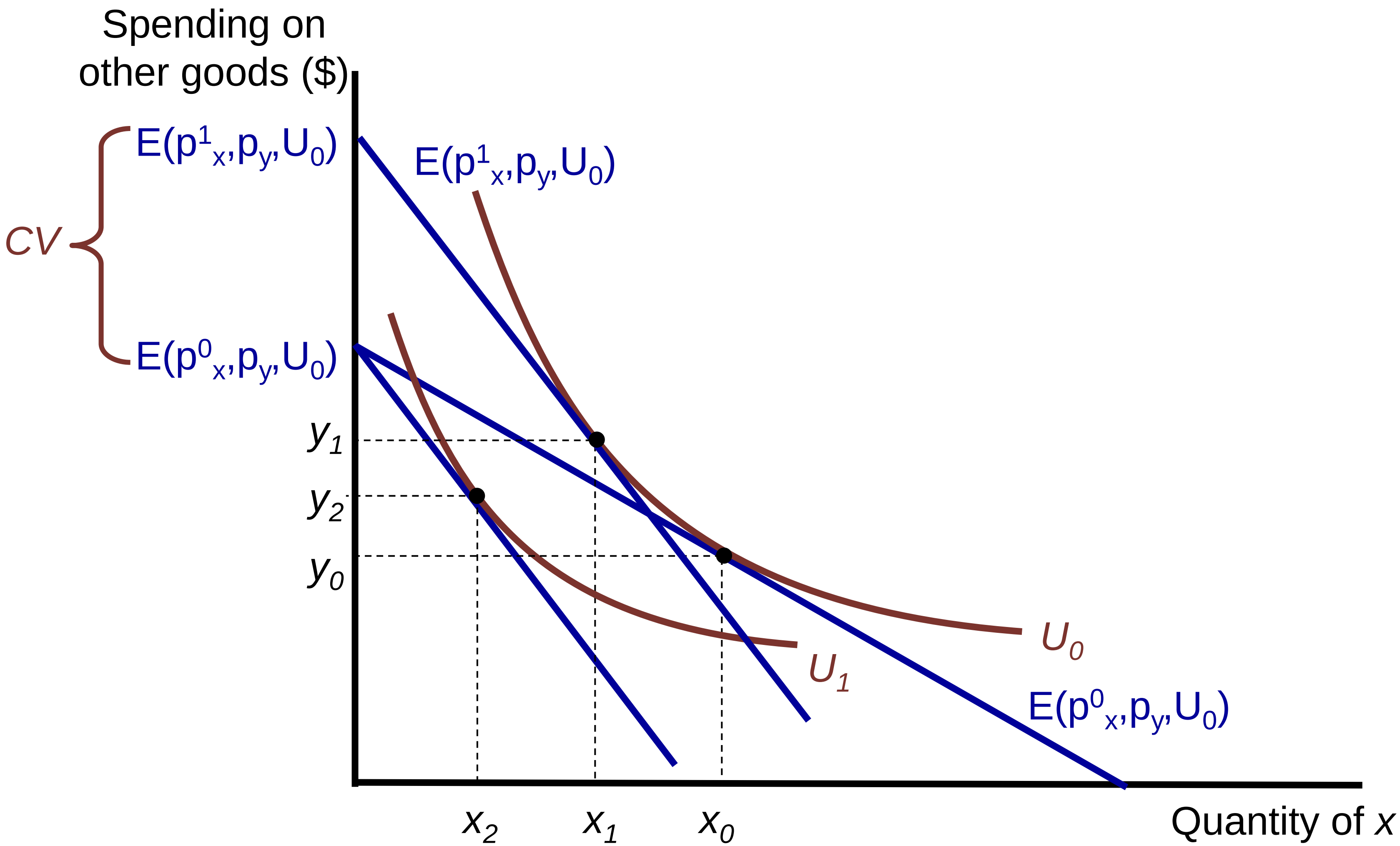
- Compensating variation (CV)
  - Money need to compensate for changes
  - $V(p^0,I^0)=V(p^1,I^0+CV)$
  - $CV = E(p_x^{\mathbf{1}},p_y,U_0) - E(p_x^{\mathbf{0}},p_y,U_0)$

$$x^c(p_x,p_y,U)=\frac{\partial E(p_x,p_y,U)}{\partial p_x}$$

$$CV=\int_{p_x^0}^{p_x^1} dE=\int_{p_x^0}^{p_x^1} x^c(p_x,p_y,U_0)dp_x$$

FIGURE 5.8 (a) Indifference curve map

Showing Compensating Variation



If the price of x increases from p^0_x to p^1_x, this person needs extra expenditures of CV to remain on the U_0 indifference curve. Integration shows that CV can also be represented by the shaded area below the compensated demand curve in panel (b).

Equivalent variation (EV)

- Equivalent variation (EV)
  - Money need avoid for changes
  - $V(p^1, I^0 + EV) = V(p^0, I^0)$
  - $EV = E(p_x^{\mathbf{1}}, p_y, U_1) - E(p_x^{\mathbf{0}}, p_y, U_1)$

Draw EV

- Draw a graph showing EV

Consumer Surplus

- Consumer surplus
  - The area below the compensated demand curve and above the market price
  - The extra benefit the person receives by being able to make market transactions at the prevailing market price

Compare CS,CV,EV

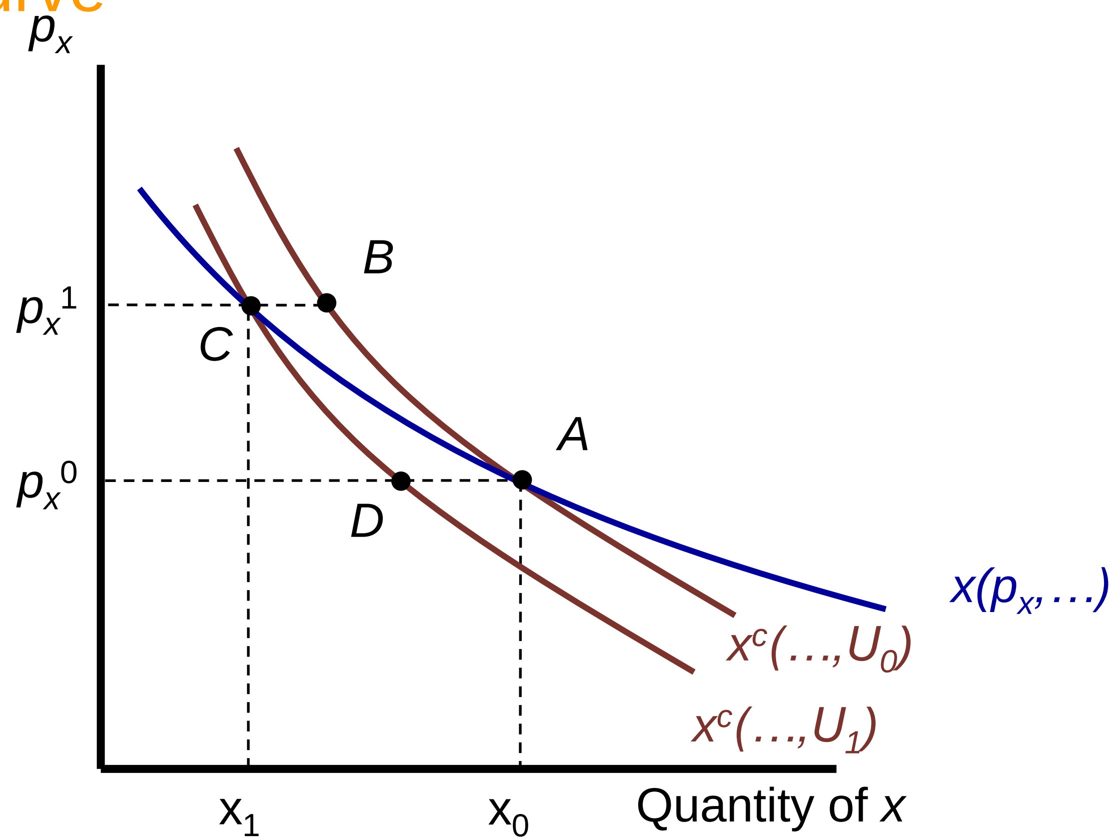
- When no income effect, they are same (why?)
- For normal good, compensated demand is steeper
  - Price increases: CV > ΔCS > EV
  - Price decreases: CV < ΔCS < EV
- What about inferior good?

Consumer Surplus

- Welfare changes and the Marshallian demand curve
- Consumer surplus
  - The area below the Marshallian demand curve and above price
    - Shows what an individual would pay for the right to make voluntary transactions at this price
  - Changes in consumer surplus measure the welfare effects of price changes

FIGURE 5.9

Welfare Effects of Price Changes and the Marshallian Demand Curve



The usual Marshallian (nominal income constant) demand curve for good x is $x(p_x, \ldots)$. Further, $x^c(\ldots, U_0)$ and $x^c(\ldots, U_1)$ denote the compensated demand curves associated with the utility levels experienced when p^0_x and p^1_x, respectively, prevail. The area to the left of $x(p_x, \ldots)$ between p^0_x and p^1_x is bounded by the similar areas to the left of the compensated demand curves. Hence for small changes in price, the area to the left of the Marshallian demand curve is a good measure of welfare loss.

- Compensated demand function for *x:*

$$x^c(p_x, p_y, V) = \frac{Vp_y^{0.5}}{p_x^{0.5}}$$

- The welfare cost of a price increase
- From p_x = \$1 to p_x = \$4 is given by

$$CV = \int_1^4 Vp_y^{0.5} p_x^{-0.5} = 2Vp_y^{0.5} p_x^{0.5} \Big|_{p_x=1}^{p_x=4}$$

- For V = 2 and p_y = 4: $CV = 2 \cdot 2 \cdot 2 \cdot (4)^{0.5} - 2 \cdot 2 \cdot 2 \cdot (1)^{0.5} = 8$
- If V=1 after the price increase: $EV = 1 \cdot 2 \cdot 2 \cdot (4)^{0.5} - 1 \cdot 2 \cdot 2 \cdot (1)^{0.5} = 4$

- Marshallian demand function:

$$x(p_x, p_y, I) = 0.5 I p_x^{-1}$$

- The welfare loss from a price increase from p_x = \$1 to p_x = \$4:

$$Loss = \int_1^4 0.5 I p_x^{-1} dp_x = 0.5 I \ln p_x \Big|_{p_x=1}^{p_x=4}$$

- If income (I) is equal to 8,

$$Loss = 4\ln(4) - 4\ln(1) = 4\ln(4) = 4(1.39) = 5.55$$

  - Compromise between the two amounts computed using the compensated demand functions

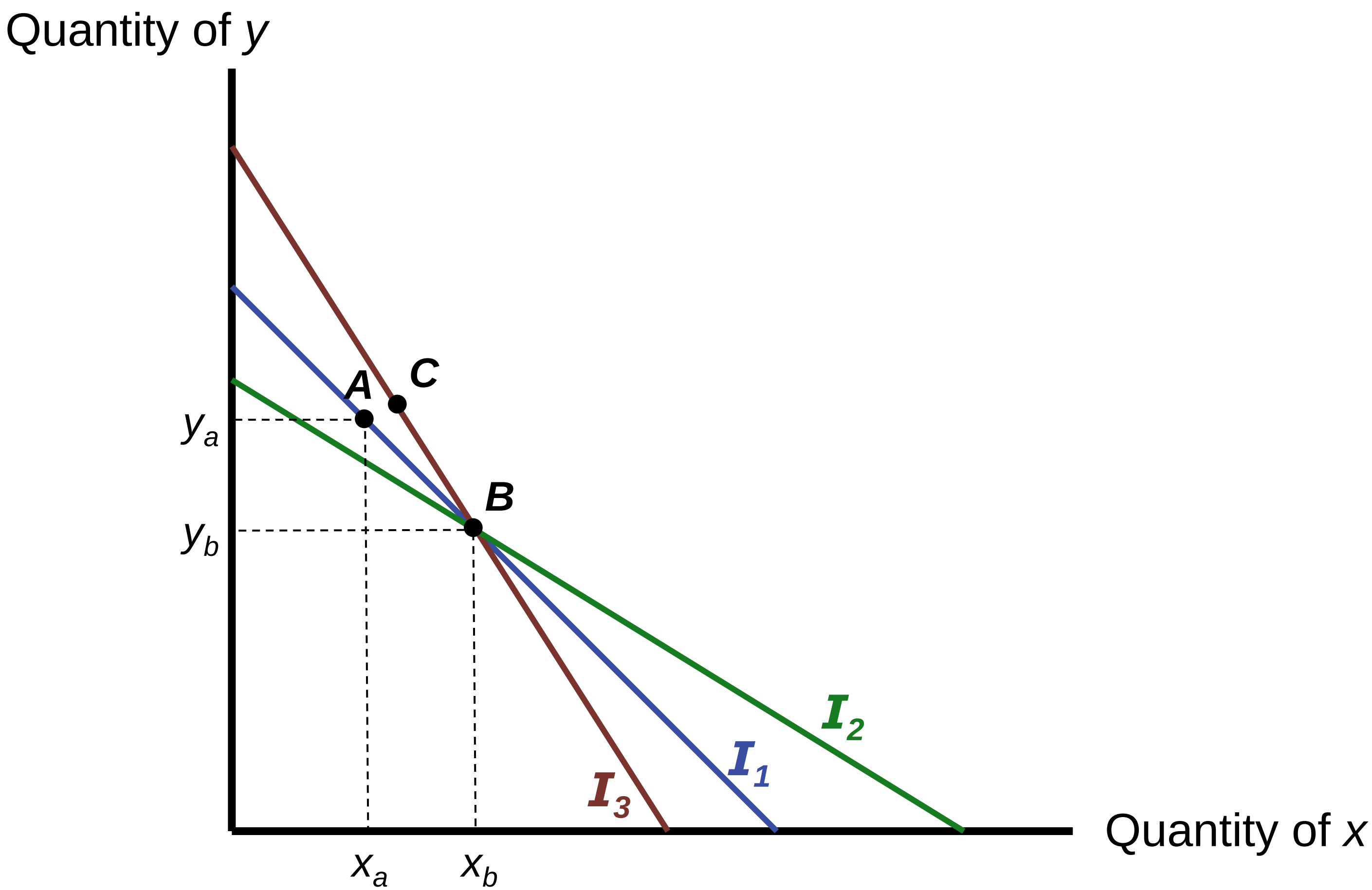
Revealed Preference and The Substitution Effect

- The theory of revealed preference
  - Proposed by Paul Samuelson in the late 1940s
  - Defines a principle of rationality based on observed behavior
  - Uses it to approximate an individual's utility function

Revealed Preference and The Substitution Effect

- Consider two bundles of goods: ***A*** and ***B***
  - If the individual can afford to purchase either bundle but chooses ***A***, we say that ***A*** had been <u>revealed preferred</u> to ***B***
  - Under any other price-income arrangement, ***B*** can never be revealed preferred to ***A***

Demonstration of the Principle of Rationality in the Theory of Revealed Preference



With income I_1 the individual can afford both points A and B. If A is selected, then A is revealed preferred to B. It would be irrational for B to be revealed preferred to A in some other price-income configuration.

Negativity of the Substitution Effect

- Suppose that an individual is indifferent between two bundles: *C* and *D*
    - p_x^C, p_y^C - prices at which bundle *C* is chosen
    - p_x^D, p_y^D - prices at which bundle *D* is chosen
  - When *C* is chosen, *D* must cost at least as much as *C*: $p_x^C x_C + p_y^C y_C \leq p_x^C x_D + p_y^C y_D$
  - When *D* is chosen, *C* must cost at least as much as *D*: $p_x^D x_D + p_y^D y_D \leq p_x^D x_C + p_y^D y_C$

Negativity of the Substitution Effect

- Rearranging: $p_x^C(x_C - x_D) + p_y^C(y_C - y_D) \leq 0$

  and $p_x^D(x_D - x_C) + p_y^D(y_D - y_C) \leq 0$

- Adding these together:

$$(p_x^C - p_x^D)(x_C - x_D) + (p_y^C - p_y^D)(y_C - y_D) \leq 0$$

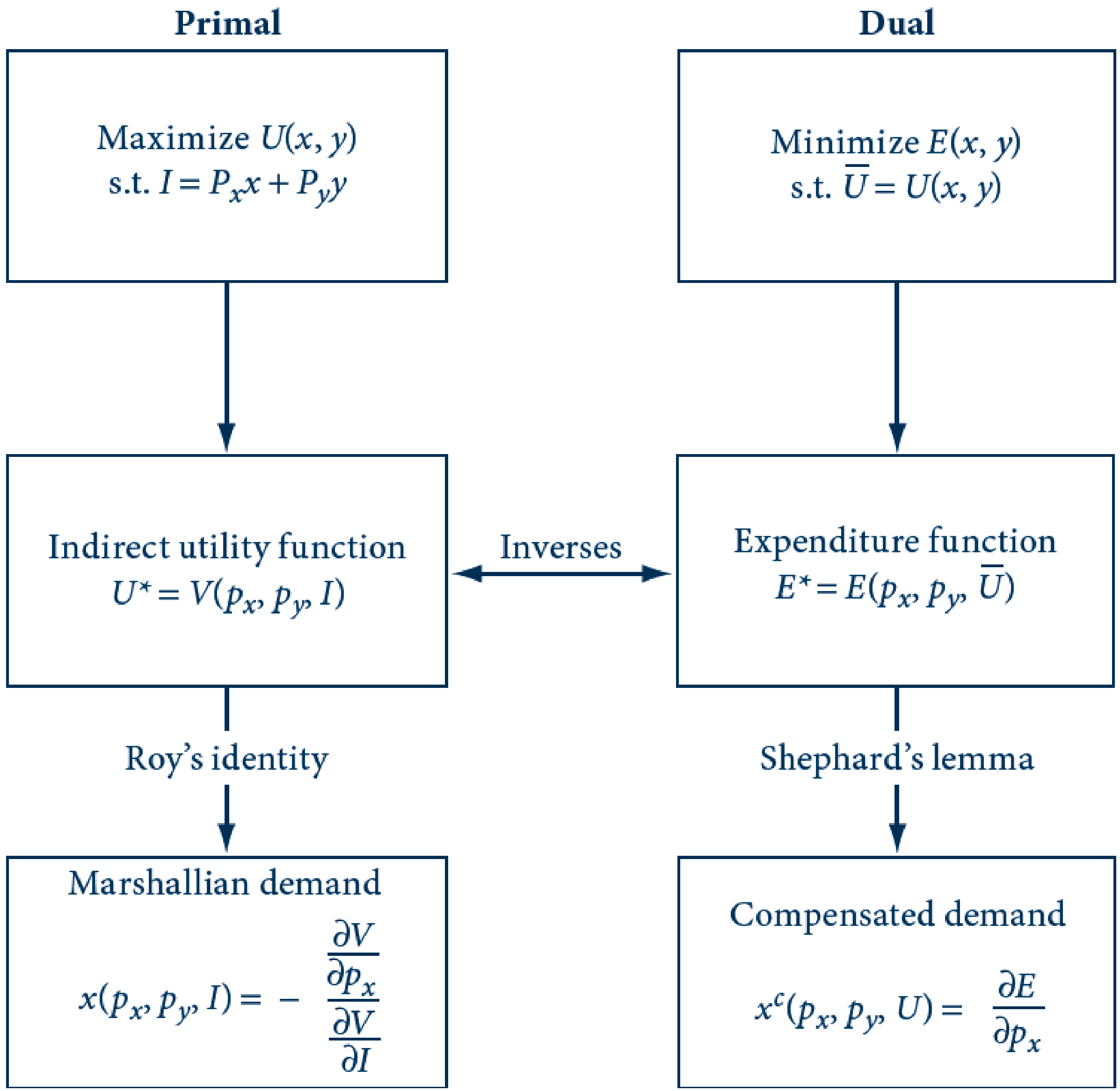
- Suppose that only the price of x changes ($p_y^C = p_y^D$): $(p_x^C - p_x^D)(x_C - x_D) \leq 0$
  - Price and quantity move in opposite direction when utility is held constant
  - The substitution effect is negative

Demand concepts

- Consumer price index (CPI)
  - "Market basket" index of the cost of living
    - Amounts that people consume of a set of goods in some base period
    - Then use current price data to compute the changing price of this market basket
  - Cost of the market basket
    - In the base period (0), $I_0 = p_x^0 x_0 + p_y^0 y_0$
    - In period 1, $I_1 = p_x^1 x_0 + p_y^1 y_0$
  - Change in the cost of living: I_1/I_0

FIGURE E5.1

Relationships among Demand Concepts

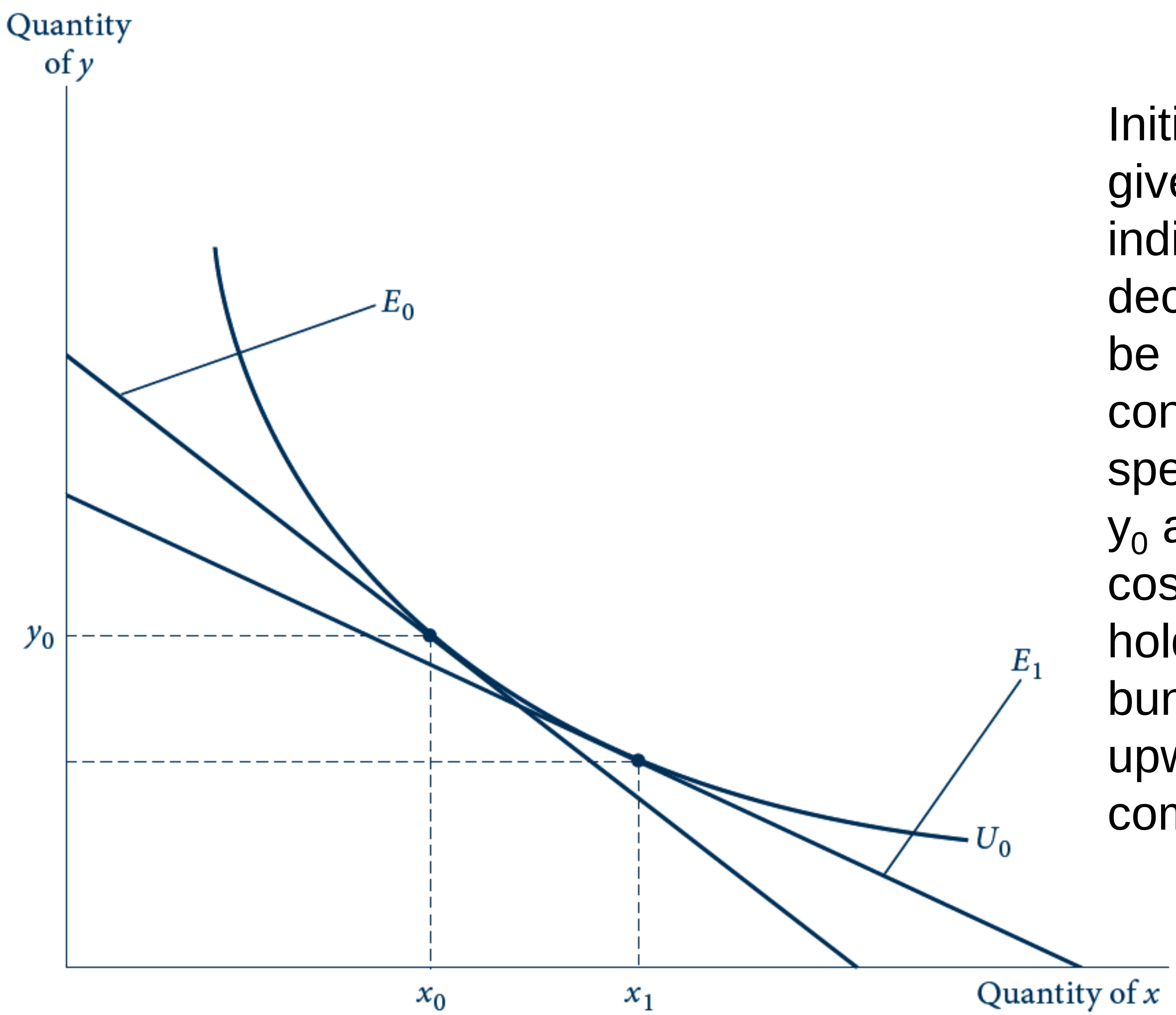


Expenditure functions & substitution bias

- Market basket price indices
  - Suffer from "substitution bias"
  - Do not permit individuals to make substitutions in the market basket
    - In response to changes in relative prices
  - Overstate the welfare losses that people incur from increasing prices

FIGURE E5.2

Substitution Bias in the CPI



Initially expenditures are given by E_0, and this individual buys x_0, y_0. If p_x/p_y decreases, utility level U_0 can be reached most cheaply by consuming x_1, y_1 and spending E_1. Purchasing x_0, y_0 at the new prices would cost more than E_1. Hence holding the consumption bundle constant imparts an upward bias to CPI-type computations.

Roy's identity and new goods bias

- When new goods are introduced
  - It takes some time for them to be integrated into the CPI
    - 15 years for cell-phones
  - Market basket indices will fail to reflect the welfare gains that people experience from using new goods

Other complaints about the CPI

- Focus on the consequences of using incorrect prices to compute the index
  - Better quality - people are made better off
    - Although this may not show up in the good's price
  - The opening of "big box" retailers
    - Reduced the prices that consumers paid for various goods
    - Including them into the CPI took several years

Exact price indices

- Two goods, x and y
  - Expenditure function $E(p_x,p_y,U)$
  - How cost of attaining the target utility level $\bar{U}$ has changed between the two periods:

$$I_{1,2}= E(p^2{}_x,p^2{}_y,\bar{U})/E(p^1{}_x,p^1{}_y,\bar{U})$$

  - Cobb–Douglas utility function, $U(x,y)=x^{\alpha}y^{1-\alpha}$
    - Expenditure function:
    - $E(p_x,p_y,U)= p_x{}^{\alpha}p_y{}^{1-\alpha}\,U/\alpha^{\alpha}(1-\alpha)^{1-\alpha}=k\, p_x{}^{\alpha}p_y{}^{1-\alpha}\,U$
    - $I_{1,2}= [(p^2{}_x)^{\alpha}(p^2{}_y)^{1-\alpha}\,]/[(p^1{}_x)^{\alpha}(p^1{}_y)^{1-\alpha}]$

Development of exact price indices

- The almost ideal demand system
  - Implies an exact price index (I) that takes a "Divisia" form

$$ln(I) = \sum_{i=1}^{n} w_i \Delta \ln p_i$$

$$\ln(I_{1,2}) = \alpha \ln p_x^2 + (1-\alpha)\ln p_y^2 - \alpha \ln p_x^1 - (1-\alpha)\ln p_y^1 =$$

$$= \alpha\Delta \ln p_x + (1-\alpha)\Delta \ln p_y$$

Demand relationship among goods

Reading

- Reading
  - Chapter 6

Change in price of other goods

- Gross complement: $\partial x/\partial p_y < 0$
- Gross substitute: $\partial x/\partial p_y > 0$
- Why? (Slutsky-type equation)

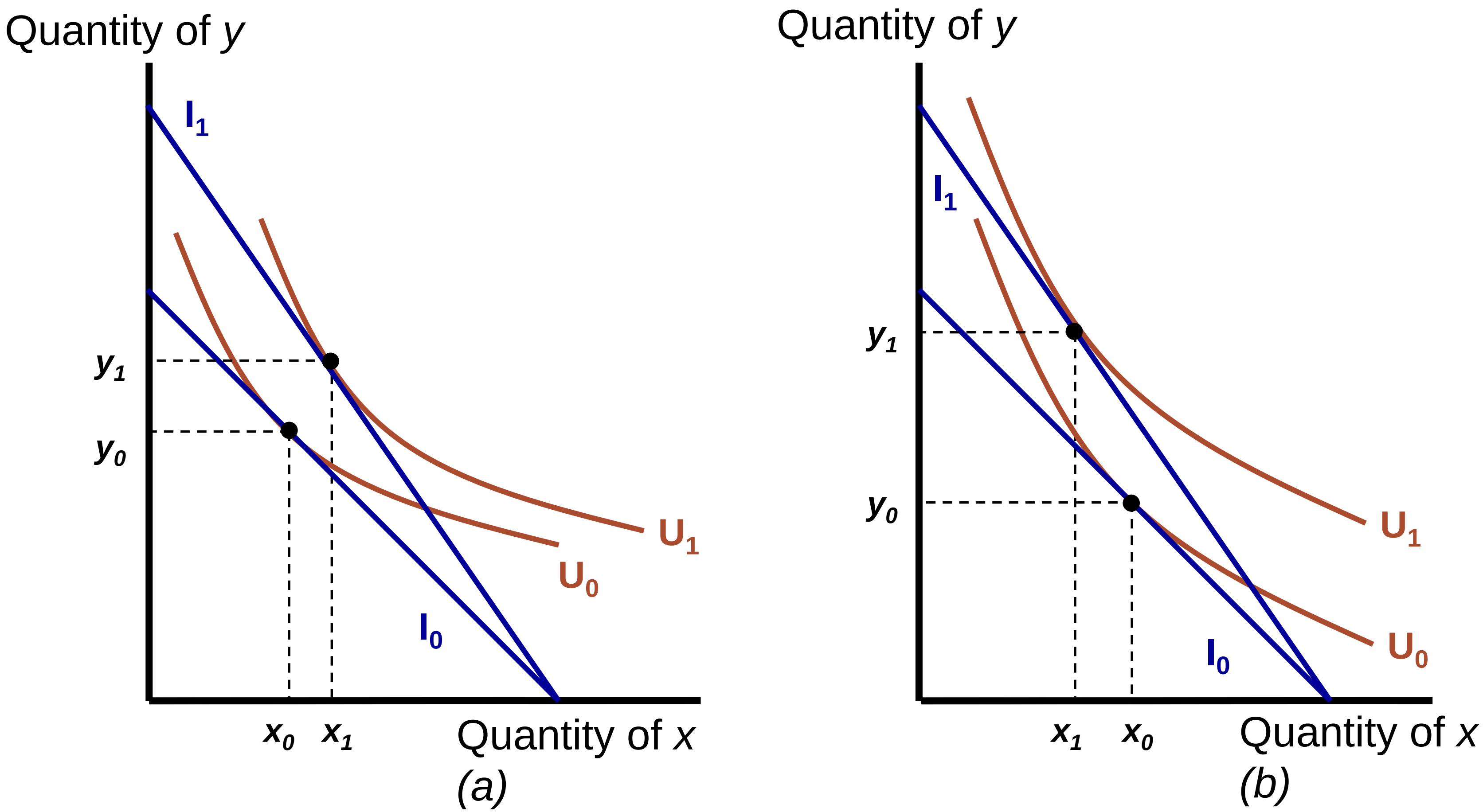
$$\frac{\partial x}{\partial p_y} = \overbrace{\underbrace{\left.\frac{\partial x}{\partial p_y}\right|_{U=\text{constant}}}_{\text{Substitution effect (+)}} \underbrace{- \; y\frac{\partial x}{\partial I}}_{\text{Income effect (-) if } x \text{ is normal}}}^{}$$

Combined effect (ambiguous)

- Elasticity term: $e_{x,p_y} = e_{x^c,p_y} - s_y e_{x,I}$

FIGURE 6.1

Differing Directions of Cross-Price Effects



In both panels, the price of y has decreased. In (a), substitution effects are small; therefore, the quantity of x consumed increases along with y. Because $\partial x/\partial p_y < 0$, x and y are gross complements. In (b), substitution effects are large; therefore, the quantity of x chosen decreases. Because $\partial x/\partial p_y > 0$, x and y would be termed gross substitutes.

EXAMPLE 6.1 Another Slutsky Decomposition for Cross-Price Effects

- The cross-price effect of a change in y prices on x purchases
  - Uncompensated demand function:

$x(p_x,p_y,I)=0.5\ I/p_x$

  - Compensated demand function:

$x^c(p_x,p_y,V)=Vp_y^{0.5}p_x^{-0.5}$

  - Marshallian demand function in this case yields:

$\partial x/\partial p_y=0$

EXAMPLE 6.1 Another Slutsky Decomposition for Cross-Price Effects

- The cross-price effect of a change in y prices on x purchases
  - Changes in the price of y do not affect x purchases
  - Because the substitution and income effects of a price change are precisely counterbalancing
- Substitution effect:

$$\left.\frac{\partial x}{\partial p_y}\right|_{U=\text{constant}} = \frac{\partial x^c}{\partial p_y} = 0.5Vp_y^{-0.5}p_x^{-0.5} = 0.25Ip_y^{-1}p_x^{-1}$$

where $V = 0.5Ip_y^{-0.5}p_x^{-0.5}$

EXAMPLE 6.1 Another Slutsky Decomposition for Cross-Price Effects

- Income effect:

$$-y\frac{\partial x}{\partial I} = -\left[0.5Ip_y^{-1}\right]\cdot\left[0.5p_x^{-1}\right] = -0.25Ip_y^{-1}p_x^{-1}$$

$$\text{where} \qquad y = 0.5Ip_y^{-1}$$

- Total effect of a change in the price of y:

$$\frac{\partial x}{\partial p_y} = 0.25Ip_y^{-1}p_x^{-1} - 0.25Ip_y^{-1}p_x^{-1} = 0$$

Many Good Cases

- Slutsky's equation

$$\frac{\partial x_i(p_1,\ldots,p_n,I)}{\partial p_j} = \left.\frac{\partial x_i}{\partial p_j}\right|_{U=\text{constant}} - x_j\frac{\partial x_i}{\partial I}$$

- In elasticity term: $e_{i,j} = e^c_{i,j} - s_j e_{i,I}$

- x_i is gross substitutes for x_j: $\partial x_i/\partial p_j>0$
- x_i is gross complement for x_j : $\partial x_i/\partial p_j<0$
- Relationship is NOT symmetric!

- Utility function, $U(x,y) = \ln x + y$
  - The Lagrangian, $\mathscr{L} = \ln x + y + \lambda(I - p_x x - p_y y)$
  - First-order conditions

$$\partial\mathscr{L}/\partial x = 1/x - \lambda p_x = 0$$

$$\partial\mathscr{L}/\partial y = 1 - \lambda p_y = 0$$

$$\partial\mathscr{L}/\partial\lambda = I - p_x x - p_y y = 0$$

  - So, $p_x x = p_y$

EXAMPLE 6.2 Asymmetry in Cross-Price Effects

- Substitution into the budget constraint
  - Solve for the Marshallian demand function for y

  $I = p_x x + p_y y = p_y + p_y y$

  - So, $y = (I - p_y)/p_y$
  - An increase in p_y - must decrease spending on y
    - Because p_x and I are unchanged, spending on x must increase
      - $\partial x/\partial p_y > 0$, gross substitutes
      - $\partial y/\partial p_x = 0$, independent

Many Good Cases

- x_i is net substitutes for x_j:

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=\text{constant}} > 0$$

- x_i is net complement for x_j :

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=\text{constant}} < 0$$

- Relationship IS symmetric! (Note: Young's Theorem)

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{U=\text{constant}} = \left.\frac{\partial x_j}{\partial p_i}\right|_{U=\text{constant}}$$

Technical Note:
Hick's second law of demand

- "most" goods must be substitutes
- Proof:

$$x_c^i(p_1,\ldots,p_n,V)$$

$$p_1\frac{\partial x_i^c}{\partial p_1}+p_2\frac{\partial x_i^c}{\partial p_2}+\ldots+p_n\frac{\partial x_i^c}{\partial p_n}=0$$

$$e_{i1}^c+e_{i2}^c+\ldots+e_{in}^c=0, \qquad e_{ii}^c\le 0$$

$$\sum_{j\neq i}e_{ij}^c\ge 0$$

Composite Good

- Consumer chooses over *n* goods
  - Demand for x_1 will depend on $p_2, p_3, \ldots, p_n$
  - If $p_2, p_3, \ldots, p_n$ move together: lump them into a single <u>composite commodity</u>, *y*
  - Let $p_2^0 \ldots p_n^0$ represent the initial prices of these other commodities
  - $y = p_2^0 x_2 + p_3^0 x_3 + \ldots + p_n^0 x_n$
- As if consumer is choosing between x_1 and y

- Suppose that an individual receives utility from three goods:
  - Food (*x*)
  - Housing services (*y*), measured in hundreds of square feet
  - Household operations (*z*), measured by electricity use
- CES utility function

$$\text{utility} = U(x, y, z) = -\frac{1}{x} - \frac{1}{y} - \frac{1}{z}$$

- Lagrangian technique
  - Can be used to calculate Marshallian demand functions for these goods as

$$x = \frac{I}{p_x + \sqrt{p_x p_y} + \sqrt{p_x p_z}}$$

$$y = \frac{I}{p_y + \sqrt{p_y p_x} + \sqrt{p_y p_z}}$$

$$z = \frac{I}{p_z + \sqrt{p_z p_x} + \sqrt{p_z p_y}}$$

EXAMPLE 6.3 Housing Costs as a Composite Commodity

- If initially $I = 100$, $p_x = 1$, $p_y = 4$, $p_z = 1$
  - The demand functions predict: $x^* = 25$, $y^* = 12.5$, $z^* = 25$
  - \$25 is spent on food and \$75 is spent on housing-related needs
- Assume that the p_y and p_z move together
  - Define the "composite commodity" housing (h)

$h = 4y + 1z$

  - Define the initial price of housing (p_h) to be 1
    - Initial quantity of housing = total spent on housing (75)

EXAMPLE 6.3 Housing Costs as a Composite Commodity

- p_y and p_z always move together
  - p_h will always be related to these prices,

  $p_h = p_z = 0.25 p_y$

- Recalculate the demand function for x as a function of I, p_x, and p_h:

$$x = \frac{I}{p_y + 3\sqrt{p_x p_h}}$$

  - If $I = 100$, $p_x = 1$, $p_y = 4$, and $p_h = 1$,

    $x^* = 25 \qquad h^* = 75$

Theory of two-stage budgeting

- Partition of goods
  - Into m nonoverlapping groups ($r = 1, m$)
- A separate budget (I_r)
  - Devoted to each category
- Demand functions for the goods within any one category
  - Depend on the prices of goods within the category
  - And on the category's budget allocation

Theory of two-stage budgeting

- Demand is given by:

$x_i(p_1,\ldots p_n,I)=x_{i\in r}(p_{i\in r},I_r),\ r=1,m$

$$V^*(p_1,\ldots p_n,I_1,\ldots,I_m)=$$

$$=\max_{x_1,\ldots,x_n}\left[U(x_1,\ldots,x_n)\ \text{s.t.}\ \sum_{i\in r} p_i x_i \le I_r, r=1,m\right]$$

and $\max\limits_{I_1,\ldots,I_m} V^*\ \text{s.t.}\ \sum_{r=1}^{m} I_r = I$

Utility-maximization problem:

$$\max_{x_1,\ldots,x_n} U(x_1,\ldots,x_n)\ \ \text{s.t.}\ \sum_{i=1}^{n} p_i x_i \le I$$

Homothetic functions and energy demand

- Assumption
  - The utility for certain subcategories of goods is homothetic and may be separated from the demand for other commodities