EC4101
Topic 1: Consumer Theory

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1 Reading

1. Snyder and Nicholoson, Chapter 3-6, Microeconomic Thoery: Basic Principles and Extensions, 11th edition, 2012

2. Jehle and Reny, Chapter 1, Advanced Microeconomic Theory, 3rd edition, 2011

2 Preference and Utility

2.1 Preference

1. **Consumption set**: $X \subseteq \mathbb{R}^n_+$: (subset of non-negative quadrant of a n-dimensional space)

   This means there are n different goods. The i-th coordinate means quantity of the i-th good.

2. **Binary relation** R on X:

   Formally, it is subset of X^2: we can use an ordered pair to represent a binary relation!

   When we say xRy, then formally it means $(x, y) \in R$.

   (Note: if R is too abstract to thing, you can read it as x is related to y in "R-way")

3. **Complete**: for any $x, y \in X$, either xRy or yRx or both.

4. **Transitive**: for any $x, y, z \in X$, if xRy and yRz, then xRz

5. **Preference:** a complete and transitive binary relation.

   We denote $\succeq$ as a preference.

   If $x \succeq y$, we read it as x is at least as good as y.

   Sometimes, we write $x \succ y$ if $x \succeq y$ but $y \not\succeq x$. (strictly preferred to)

   We also write $x \sim y$ if $x \succeq y$ and $y \succeq x$. (indifferent to)

2.2 Property of Preference

1. **monotonic**: for all $x, y \in X$ if $x_i \geq y_i$ for all i, then $x \succeq y$

2. **strict monotonic**: for all $x, y \in X$ if $x_i \geq y_i$ for all i and for some $x_i > y_i$, then $x \succ y$

3. **continuous**: $B(x)$ and $W(x)$ are closed where $B(x) \equiv \{y \in X : y \succeq x\}$ be the no-worse set of x and $W(x) \equiv \{y \in X : x \succeq y\}$ be the no better set of x.

4. **convex**: for all $x, y \in X$, if $x \succeq y$, then $\lambda x + (1 - \lambda) y \succeq y$ for all $0 \leq \lambda \leq 1$.

5. **strictly convex**: for all $x, y \in X$, if $x \succeq y$ and $x \neq y$, then $\lambda x + (1 - \lambda) y \succ y$ for all $0 \leq \lambda \leq 1$.

6. **quasi-concave:** $B(x)$ is a convex set

2.3 Indifference curve

1. It is graphical representation of preference

2. Property:

   (a) Under monotonicity, northwest direction is better (why?)

   (b) Indifference curve does not cross (why?)

2.4 Utility

1. A preference $\succeq$ is represented by a **utility function** U if for all $x, y \in X$

$$x \succeq y \text{ if and only if } U(x) \geq U(y)$$

2. Utility function assigns a number for every consumption bundle: more preferred bundle gets larger number.

3. Not every preference is represented by a utility function (Example: lexicographic)

4. Ordinal utility: only order matters

5. Marginal rate of substitution of good 1 for good 2:

$$MRS_{12} = -\left.\frac{dx_2}{dx_1}\right|_{U(x)=u} = \frac{\partial U(x)/\partial x_1}{\partial U(x)/\partial x_2}$$

6. Technical note:

   (a) Continuous and strictly montonic preference implies existence of utility function.

   (b) **Quasi-concave utility**: for all $x, y \in X$, $U(\lambda x+(1-\lambda) y) \geq \min \{U(x), U(y)\}$ for all $0 \leq \lambda \leq 1$

   (c) Preference is (strictly) convex if and only if utility function is (strictly) quasi-concave

   (d) If utility function is differentiable and preference is (strictly) convex, then we have diminishing MRS.

   (e) Homothetic preference: MRS between two goods depends **only** on the ratio of quantities of two goods

Specific utility functions:

1. Perfect substituties: $U(x, y) = \alpha x + \beta y$ for some $\alpha > 0$, $\beta > 0$

2. Perfect complements: $U(x, y) = \min\{\alpha x, \beta y\}$ for some $\alpha > 0$, $\beta > 0$

3. Cobb-Douglas: $U(x, y) = x^{\alpha} y^{1-\alpha}$ for some $0 < \alpha < 1$

4. Constant elasticity of substitution (CES): frequently used in macro/IO

$$U(x,y) = \frac{x^\delta}{\delta} + \frac{y^\delta}{\delta}$$

   (a) See appendix for discussion on **elasticity of substitution**

   (b) This is general case for the above three special functions:

      i. Perfect substitutes: $\delta = 1$

      ii. Cobb-Douglas: $\delta = 0$

      iii. Perfect Complement: $\delta = -\infty$

3 Utility Maximization and Choice

1. **Utility Maximization problem**:(UMP)

$$\max_{x_1, x_2, \ldots, x_n} U(x_1, x_2, \ldots, x_n)$$

subject to budget constraint

$$p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = I$$

Lagrangian:

$$\mathcal{L} = U(x_1, x_2, \ldots, x_n) + \lambda \left(I - p_1 x_1 - p_2 x_2 - \cdots - p_n x_n \right)$$

where first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i} &= 0 \text{ for } i = 1, \ldots, n \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \end{aligned}$$

Interior solution:

$$MRS_{ij} = \frac{\partial U / \partial x_i}{\partial U / \partial x_j} = \frac{p_i}{p_j}$$

Economic interpration of λ and p_i.

$$\begin{aligned}\lambda &= \frac{\partial U/\partial x_i}{p_i} \\ p_i &= \frac{\partial U/\partial x_i}{\lambda}\end{aligned}$$

Corner solution: (Economic meaning?)

$$p_i > \frac{\partial U/\partial x_i}{\lambda}$$

2. **Marshallian demand function**: The optimal solution of UMP

$$\begin{aligned}x_1^* &= x_1(p_1, \ldots, p_n, I) \\ &\ldots \\ x_n^* &= x_n(p_1, \ldots, p_n, I)\end{aligned}$$

3. Property of Marshallian demand function:

   (a) Homogeneous of degree zero in prices and income:

   $$\begin{aligned}x_1(\lambda p_1 \ldots, \lambda p_n, \lambda I) &= x_1(p_1 \ldots, p_n, I) \\ &\ldots \\ x_n(\lambda p_1 \ldots, \lambda p_n, \lambda I) &= x_n(p_1 \ldots, p_n, I)\end{aligned}$$

   for any $\lambda > 0$

   (b) Other properites will be covered in next section.

4. **Indirect utility function**:

   (a) express optimal level of utility in terms of prices and income.

   (b) the optimal-value function of UMP: by putting the marshallian demand function into the utility function

$$
\begin{aligned}
& U(x_1^*, x_2^*, \ldots, x_n^*) \\
= \; & U(x_1(p_1, \ldots, p_n, I), x_2(p_1, \ldots, p_n, I), \ldots, x_n(p_1, \ldots, p_n, I)) \\
= \; & V(p_1 \ldots, p_n, I)
\end{aligned}
$$

5. Property of indirect utility function:

   (a) Non-decreasing in income: $\frac{\partial V}{\partial I} \geq 0$

   (b) Non-increasing in prices: $\frac{\partial V}{\partial p_i} \leq 0$ for every good i

   (c) Homogeneous of degree zero in prices and income: $V(\lambda p_1 \ldots, \lambda p_n, \lambda I) = V(p_1 \ldots, p_n, I)$ for any $\lambda > 0$

6. Application of indirect utility: lump-sum principle. lump-sum tax is better than tax on specific good.

7. **Expenditure minimization problem** (EMP; dual problem to the UMP)

$$
\min_{x_1, x_2, \ldots, x_n} p_1 x_1 + p_2 x_2 + \cdots + p_n x_n
$$

subject to budget constraint

$$
U(x_1, x_2, \ldots, x_n) = \bar{U}
$$

Lagrangian:

$$
\mathcal{L} = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n + \lambda \left(\bar{U} - U(x_1, x_2, \ldots, x_n) \right)
$$

where first-order conditions are

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_i} &= 0 \text{ for } i = 1, \ldots, n \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0
\end{aligned}
$$

8. **Hicksian demand function:** the optimal solution to EMP

$$\begin{aligned} x_1^* &= x_1^c(p_1, \ldots, p_n, \bar{U}) \\ &\ldots \\ x_n^* &= x_n^c(p_1, \ldots, p_n, \bar{U}) \end{aligned}$$

9. Properties of expenditure function:

   (a) Homogeneous of degree one in prices: $x^c(\lambda p_1 \ldots, \lambda p_n, U) = x^c(p_1 \ldots, p_n, U)$ for any $\lambda > 0$

   (b) Other properites will be covered in next section.

10. **Expenditure function:**

    (a) express minimum expenditure required given in terms of prices and utility level

    (b) the optimal-value function of EMP: by putting the Hicksian demand function into the expenditure

$$\begin{aligned} & p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \\ = \ & p_1 x_1^c(p_1, \ldots, p_n, \bar{U}) + p_2 x_2^c(p_1, \ldots, p_n, \bar{U}) + \cdots + p_n x_n^c(p_1, \ldots, p_n, \bar{U}) \\ = \ & E(p_1, \ldots, p_n \bar{U}) \end{aligned}$$

11. Properties of expenditure function:

    (a) Homogeneous of degree one in prices: $E(\lambda p_1 \ldots, \lambda p_n, \lambda I) = V(p_1 \ldots, p_n, I)$ for some $\lambda > 0$

    (b) Non-decreasing in prices: $\dfrac{\partial E}{\partial p_i} \geq 0$ for every good i

    (c) Concave in prices: (why?)

    (d) Other properites will be covered in next section.

12. Class exercise: Show expenditure function of $U(x, y) = \sqrt{xy}$ is $E(p_x, p_y, U) = 2\sqrt{p_x p_y} U$

4 Income and substitution effects

4.1 Elasticity

1. Marshallian Demand Elasticities

   (a) **(Own) Price elasticity of demand**

   $$e_{i,p_i} = \frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i}$$

   Note that
   $$\frac{\partial p_i x_i}{\partial p_i} = x_i + p_i \frac{\partial x_i}{\partial p_i} = x_i\left[1 + e_{i,p_i}\right]$$

   Elastic demand: $e_{i,p_i} < -1 \Leftrightarrow \frac{\partial p_i x_i}{\partial p_i} > 0$

   Inelastic demand: $e_{i,p_i} > -1 \Leftrightarrow \frac{\partial p_i x_i}{\partial p_i} < 0$

   Unit elastic demand: $e_{i,p_i} = -1 \Leftrightarrow \frac{\partial p_i x_i}{\partial p_i} = 0$

   (b) Cross-price elasticity of demand

   $$e_{i,pj} = \frac{\partial x_i}{\partial p_j}\frac{p_j}{x_i}$$

   Good i is gross complement for j: $e_{i,p_j} < 0$

   Good i is gross substitute for j: $e_{i,p_i} > 0$

   (c) Income elasticities of demand

   $$e_{i,I} = \frac{\partial x_i}{\partial I}\frac{I}{x_i}$$

2. Compensated Demand Elasticities

   (a) **(Own) Price elasticity of compensated demand**

   $$e^c_{i,p_i} = \frac{\partial x^c_i}{\partial p_i}\frac{p_i}{x^c_i}$$

   Always non-negative

(b) **Cross-price elasticity of compensated demand**

$$e^c_{i,p_j} = \frac{\partial x^c_i}{\partial p_j}\frac{p_j}{x^c_i}$$

Good i is net complement for j: $e^c_{i,p_j} < 0$

Good i is net substitute for j: $e^c_{i,p_j} > 0$

(c) **Income elasticity of demand**

$$e^c_{i,I} = \frac{\partial x^c_i}{\partial I}\frac{I}{x^c_i}$$

3. Relationship between elasticity is covered in Appendix.

4.2 Two important expressions

1. **Roy's Identity**

   Since V is optimal-value function for langrangian minimization

   $$\mathcal{L} = U(x_1, x_2, \ldots, x_n) + \lambda(I - p_1x_1 - p_2x_2 - \cdots - p_nx_n)$$

   Hence, envelop theorem implies

   $$\begin{aligned}\frac{\partial V}{\partial p_i} &= -\lambda x_i \\ \frac{\partial V}{\partial I} &= \lambda\end{aligned}$$

   so that

   $$x_i = -\frac{\frac{\partial V}{\partial p_i}}{\frac{\partial V}{\partial I}}$$

2. **Shephard's Lemma**

   Since E is optimal-value function for langrangian minimization

   $$\mathcal{L} = p_1x_1 + p_2x_2 + \cdots + p_nx_n + \lambda(U(x_1, x_2, \ldots, x_n) - \bar{U})$$

Hence, envelop theorem implies

$$\frac{\partial E}{\partial p_i} = x_i^c$$

4.3 Changes of Income and Prices

1. Change in income

   (a) **Normal good**: $\dfrac{\partial x_i}{\partial I} \geq 0$

   (b) **Inferior good**: $\dfrac{\partial x_i}{\partial I} < 0$

2. Change in own price: substitution effect and income effect

   (a) graphical analysis

   (b) mathematical analysis (**Slutsky's equation:** See appendix for proof)

   $$\frac{\partial x}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}} - x\frac{\partial x}{\partial I}$$

3. Change in other prices

   (a) Slutsky's style equation:

   $$\frac{\partial x_i}{\partial p_j} = \left.\frac{\partial x_i}{\partial p_j}\right|_{U=\text{constant}} - x_j\frac{\partial x_i}{\partial I}$$

   (b) Good i is **gross complement** for j: $\dfrac{dx_i}{dp_j} < 0$

   Good i is **gross substitute** for j: $\dfrac{dx_i}{dp_j} > 0$

   Note that: gross susbstitute/complement is not symmetric

   (c) Good i is **net complement** for j: $\dfrac{dx_i^c}{dp_j} < 0$

   Good i is **net substitute** for j: $\dfrac{dx_i^c}{dp_j} > 0$

   Note that:

   i. net susbstitute/complement is symmetric (Note: Shephard's lemma and Young's theorem)

ii. In two-good world, two goods must be net substitute to each other

iii. For visual illustration of net complements in three-good world, check Paul A. Samuelson (1974) "Complementarity: An Essay on The 40th Anniversary of the Hicks-Allen Revolution in Demand Theory" in Journal of Economic Literature , Vol. 12, No. 4 (Dec., 1974), pp. 1255-1289.

iv. **Hick's second law**: almost all are net substitutes. See appendix for details.

4. **Giffen's paradox**. positive relationship between price and quantity demanded when incrome effect dominates substitution effect for inferior good

4.4 Consumer Welfare of price change

1. Measure of Consumer Welfare of price change

   Suppose price in good i changes from p_i^0 to p_i^1

   (a) **Conumser surplus** (CS): area under marshallian demand curve

   $$\Delta CS = \int_{p_i^0}^{p_i^1} x_i(p_i, \ldots, I)dp_i$$

   (b) **Compensating variation** (CV): change in income to needed to be as happy as before

   $$V(p_i^0, \ldots, I) = V(p_i^1, \ldots, I + CV)$$

   or

   $$\begin{aligned} CV &= E(p_i^1, \ldots, U^0) - E(p_i^0, \ldots, U^0) \\ &= \int_{p_i^0}^{p_i^1} dE(p_i, \ldots, U^0) \\ &= \int_{p_i^0}^{p_i^1} x_i^h(p_i, \ldots, U^0)dp_i \text{ (by Shephard's Lemma)} \end{aligned}$$

   (c) **Equivalent variation** (EV): change in income needed to avoid change

   $$V(p_i^0, \ldots, I - EV) = V(p_i^1, \ldots, I)$$

or

$$\begin{aligned} EV &= E(p_i^1, \ldots, U^1) - E(p_i^0, \ldots, U^1) \\ &= \int_{p_i^0}^{p_i^1} dE(p_i, \ldots, U^1) \\ &= \int_{p_i^0}^{p_i^1} x_i^h(p_i, \ldots, U^1) dp_i \text{ (by Shephard's Lemma)} \end{aligned}$$

2. Comparison between ΔCS, CV, and EV

   (a) When there is no income effect, all three are the same.

   (b) For normal goods,

$$\begin{array}{rl} \text{Price increase:} & CV > \Delta CS > EV \\ \text{Price decrease:} & CV < \Delta CS < EV \end{array}$$

   (c) For inferior goods

$$\begin{array}{rl} \text{Price increase:} & CV < \Delta CS < EV \\ \text{Price decrease:} & CV > \Delta CS > EV \end{array}$$

4.5 Revealed Preference

1. (a) x is **revealed preferred** to y: if x and y are feasible under some prices, and x is chosen

   (b) **Weak axiom of revealed preference**: Suppose x is chosen under prices p and $\hat{x}$ is chosen under $\hat{p}$. If $p_1x_1 + \cdots + p_nx_n \geq p_1\hat{x}_1 + \cdots + p_n\hat{x}_n$, then $\hat{p}_1x_1 + \cdots + \hat{p}_nx_n > \hat{p}_1\hat{x}_1 + \cdots + \hat{p}_n\hat{x}_n$. In words, if x is revelaed preferred to $\hat{x}$, then $\hat{x}$ is never revealed preferred to x.

   (c) Non-positive substitution effect: compensated demand curve is downward sloping without quasi-concave preference (only need Weak axiom of revealed preference)

      i. Special case proof: Suppose a consumer is indifferent on (x_1, x_2) and $(\hat{x}_1, \hat{x}_2)$. Suppose (x_1, x_2) is chosen under prices (p_1, p_2) and $(\hat{x}_1, \hat{x}_2)$ is chosen under prices $(\hat{p}_1, p_2)$. Then

$$\begin{aligned} p_1 x_1 + p_2 x_2 &\leq p_1 \hat{x}_1 + p_2 \hat{x}_2 \\ \hat{p}_1 \hat{x}_1 + p_2 \hat{x}_2 &\leq \hat{p}_1 x_1 + p_2 x_2 \end{aligned}$$

Summing up we have

$$\begin{aligned} p_1 x_1 + \hat{p}_1 \hat{x}_1 &\leq p_1 \hat{x}_1 + \hat{p}_1 x_1 \\ p_1 (x_1 - \hat{x}_1) + \hat{p}_1 (\hat{x}_1 - x_1) &\leq 0 \end{aligned}$$

or

$$(p_1 - \hat{p}_1)(\hat{x}_1 - x_1) \leq 0$$

which is non-positive substitution effect! (without any assumption on preference!)

ii. General case. See appendix.

5 Appendix.

1. **Elasticity of substitution**: relative change in the ratio of goods in respondence to a change in the ratio of prices

$$\varepsilon_{x,y}^{sub} = \frac{d \ln (x/y)}{d \ln (p_x/p_y)}$$

For CES, we have at optimal

$$\frac{dx}{dy} = -\frac{U_y}{U_x} = -\left(\frac{y}{x}\right)^{\delta - 1} = -\frac{p_x}{p_y}$$

so that

$$\frac{x}{y} = \left(\frac{p_x}{p_y}\right)^{\frac{1}{1-\delta}}$$

$$
\begin{aligned}
\varepsilon^{sub} &= \frac{d\ln(x/y)}{d\ln(p_x/p_y)} = \frac{\frac{d(x/y)}{x/y}}{\frac{d(p_x/p_y)}{p_x/p_y}} = \frac{p_x/p_y}{x/y}\frac{d(x/y)}{d(p_x/p_y)} \\
&= \left(\frac{p_x}{p_y}\right)^{1-\frac{1}{1-\delta}}\left(\frac{1}{1-\delta}\right)\left(\frac{p_x}{p_y}\right)^{\frac{1}{1-\delta}-1} = \frac{1}{1-\delta}
\end{aligned}
$$

2. Constrained Optimization: **Lagrangian Method**

To solve

$$\max_{x_1,x_2,\dots,x_n} f(x_1,x_2,\dots,x_n)$$

subject to a constraint

$$g(x_1,x_2,\dots,x_n) = 0$$

We can write Lagrangian expression

$$\mathcal{L} = f(x_1,x_2,\dots,x_n) + \lambda g(x_1,x_2,\dots,x_n)$$

The optimal interior solution will satisfy the following conditions:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{\partial f}{\partial x_1} + \lambda\frac{\partial g}{\partial x_1} = 0 \\
&\dots \\
\frac{\partial \mathcal{L}}{\partial x_n} &= \frac{\partial f}{\partial x_n} + \lambda\frac{\partial g}{\partial x_n} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= g = 0
\end{aligned}
$$

Note that:

1. We require f to be a differentiable functions.

2. This only work for interior solution. You might need to check corner solution

3. **Simple envelope theorem**

   Maximization of $f(x;\theta)$ where θ is a parameter. Optimal-value function is then

   $$f^*(\theta) = \max_x f(x;\theta)$$

   and the optimal-solution function is $x^*(\theta)$

Envelope theorem states:

$$\frac{df^*(\theta)}{d\theta} = \frac{\partial f(x;\theta)}{\partial \theta}\bigg|_{x=x^*(\theta)}$$

Note. This also works for minimization problem.

4. **General envelope theorem**

   Maximization of $f(x;\theta)$ subject to $g(x;\theta) = 0$ where θ is a parameter.

   The Lagrangian is

   $$\mathcal{L}(x;\theta) = f(x;\theta) + \lambda g(x;\theta)$$

   Optimal-value function is then

   $$f^*(\theta) = \max_{x \in \{g(x;\theta)=0\}} f(x;\theta)$$

   and the optimal-solution function is $x^*(\theta)$

   Envelope theorem states:

   $$\frac{df^*(\theta)}{d\theta} = \frac{\partial \mathcal{L}(x;\theta)}{\partial \theta}\bigg|_{x=x^*(\theta),\lambda=\lambda(\theta)}$$

   Note. This also works for minimization problem.

5. **Euler's Theorem**

   If f is homogenous in degree k, then

   $$kf(x_1, x_2, \ldots, x_n) = x_1 \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_1} + \cdots + x_n \frac{\partial f(x_1, x_2, \ldots, x_n)}{\partial x_n}$$

   Proof.

   Recall: homogenous function of degree k implies for all $\lambda > 0$,

   $$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, x_2, \ldots, x_n).$$

Differentiate both sides with respect to λ,

$$k\lambda^{k-1} f(x_1, x_2, \ldots, x_n) = x_1 \frac{\partial f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)}{\partial x_1} + \cdots + x_n \frac{\partial f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n)}{\partial x_n}$$

Consider the case $\lambda = 1$. Then we are done.

6. **Slutsky's equation**

Since EMP and UMP give the same outcome we have

$$x^c(p_x, p_y, U) = x\left(p_x, p_y, E(p_x, p_y, U)\right)$$

so that partially differentiation gives

$$\frac{\partial x^c}{\partial p_x} = \frac{\partial x}{\partial p_x} + \frac{\partial x}{\partial E}\frac{\partial E}{\partial p_x}$$

or

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - \frac{\partial x}{\partial E}\frac{\partial E}{\partial p_x}$$

Shephard's Lemma implies

$$\frac{\partial E}{\partial p_x} = x^c$$

so that

$$\begin{aligned}
\frac{\partial x}{\partial p_x} &= \frac{\partial x^c}{\partial p_x} - x^c \frac{\partial x}{\partial E} \\
&= \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}} - x^c \frac{\partial x}{\partial E} \\
&= \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}} - x^c \frac{\partial x}{\partial I}
\end{aligned}$$

Note that at optimum we have

$$\begin{aligned}
x^c(p_x, p_y, U) &= x^c(p_x, p_y, V(p_x, p_y, I)) \\
&= x(p_x, p_y, I)
\end{aligned}$$

Then the slutsky euqation becomes

$$\frac{\partial x}{\partial p_x} = \left.\frac{\partial x}{\partial p_x}\right|_{U=\text{constant}} - x\frac{\partial x}{\partial I}$$

7. Elasticity Relationship

   (a) Homogeneous of degree zero in prices and income: By Euler's theorem implies for

   $$p_1\frac{\partial x_i}{\partial p_1} + p_2\frac{\partial x_i}{\partial p_2} + \cdots + p_n\frac{\partial x_i}{\partial p_n} + I\frac{\partial x_i}{\partial I} = 0 \text{ for all } i \in N$$

   or divide x_i on both sides

   $$\begin{aligned} \frac{p_1}{x_i}\frac{\partial x_i}{\partial p_1} + \frac{p_2}{x_i}\frac{\partial x_i}{\partial p_2} + \cdots + \frac{p_n}{x_i}\frac{\partial x_i}{\partial p_n} + \frac{I}{x_i}\frac{\partial x_i}{\partial I} &= 0 \text{ for all } i \in N \\ e_{i,p_1} + \cdots + e_{i,p_n} + e_{i,I} &= 0 \end{aligned}$$

   (b) **Engel aggregation**: Total differentiate I with respect to budget constraint

   $$\begin{aligned} 1 &= p_1\frac{\partial x_1}{\partial I} + \cdots + p_n\frac{\partial x_n}{\partial I} \\ 1 &= \frac{p_1 x_1}{I}\frac{I}{x_1}\frac{\partial x_1}{\partial I} + \cdots + \frac{p_n x_n}{I}\frac{I}{x_n}\frac{\partial x_n}{\partial I} \end{aligned}$$

   so that

   $$1 = s_1 e_{1,I} + \cdots + s_n e_{n,I}$$

   (c) **Cournot aggregation**:Total differentiate p_i with respect to budget constraint

   $$0 = p_1\frac{\partial x_1}{\partial p_i} + \cdots + x_i + p_i\frac{\partial x_i}{\partial p_i} + \cdots + p_n\frac{\partial x_n}{\partial p_i}$$

   Then multiply both sides by p_i/I

   $$\begin{aligned} 0 &= p_1\frac{\partial x_1}{\partial p_i}\frac{p_i}{I} + \cdots + x_i\frac{p_{x_i}}{I} + p_i\frac{\partial x_i}{\partial p_i}\frac{p_{x_i}}{I} + \cdots + p_n\frac{\partial x_n}{\partial p_i}\frac{p_i}{I} \\ 0 &= \frac{p_1 x_1}{I}\frac{\partial x_1}{\partial p_i}\frac{p_i}{x_1} + \cdots + x_i\frac{p_i}{I} + \frac{p_i x_i}{I}\frac{\partial x_i}{\partial p_i}\frac{p_i}{x_i} + \cdots \frac{p_n x_n}{I}\frac{\partial x_n}{\partial p_i}\frac{p_i}{x_n} \end{aligned}$$

   so that

   $$0 = s_1 + s_1 e_{1,p_i} + \cdots\cdots + s_n e_{n,p_i}$$

(d) Compensated and Uncompensated Price Elasticities

Recall Slutsky's equation

$$\frac{\partial x}{\partial p_x} = \frac{\partial x^c}{\partial p_x} - x\frac{\partial x}{\partial I}$$

Multiple both sides by p_x/x

$$\begin{aligned} e_{x,p_x} &= \frac{p_x}{x}\frac{\partial x}{\partial p_x} \\ &= \frac{p_x}{x}\frac{\partial x^c}{\partial p_x} - \frac{p_x}{x}x\frac{\partial x}{\partial I} \\ &= \frac{p_x}{x^c}\frac{\partial x^c}{\partial p_x} - \frac{p_x x}{I}\frac{I}{x}\frac{\partial x}{\partial I} \\ &= e^c_{x,p_x} - s_x e_{x,I} \end{aligned}$$

So they are similar if (i) s_x is small or (ii) $e_{x,I}$ is small

8. **Hick's second law**

Since compensated demand is homogeneous in degree zero in all prices, by Euler's Theorem, we have

$$p_1\frac{\partial x_i^c}{\partial p_1} + p_2\frac{\partial x_i^c}{\partial p_2} + \cdots + p_n\frac{\partial x_i^c}{\partial p_n} = 0$$

so that divding by x_i^c, we have

$$e^c_{i,p_1} + e^c_{i,p_2} + \cdots + e^c_{i,p_n} = 0$$

Since we know that $e^c_{i,p_i} \leq 0$, we have

$$\sum\nolimits_{j\neq i} e^c_{i,p_j} \leq 0$$

9. **Walra's Law**.

Under monotonic preference, optimal consumption bundle is on the budget curve.

$$p_1x_1 + \cdots + p_nx_n = I$$

10. Downward sloping compensated demand

Suppose x is chosen under prices p and income I. Consider some prices $\hat{p}$. Let

$\hat{I} = \hat{p}_1 x_1 + \hat{p}_2 x_2 + \cdots + \hat{p}_n x_n$. Then either

$$\begin{aligned} x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) &= x_i(p_1, \ldots, p_n, I) \text{ for all } i \\ \text{or } \sum\nolimits_i (\hat{p}_i - p_i)\left(x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) - x_i(p_1, \ldots, p_n, I)\right) &< 0 \end{aligned}$$

Proof. Assume $x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) \neq x_i(p_1, \ldots, p_n, I)$ for some i.

$$\begin{aligned} & \sum\nolimits_i (\hat{p}_i - p_i)\left(x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) - x_i(p_1, \ldots, p_n, I)\right) \\ = & \sum\nolimits_i \left[\hat{p}_i x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) - \hat{p}_i x_i(p_1, \ldots, p_n, I) - p_i x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) + p_i x_i(p_1, \ldots, p_n, I)\right] \\ = & \sum\nolimits_i \left[\hat{I} - \hat{I} - p_i x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I}) + I\right] \text{ (by Walra's law)} \\ = & \sum\nolimits_i \left[I - p_i x_i(\hat{p}_1, \ldots, \hat{p}_n, \hat{I})\right] \leq 0 \text{ (by weak axiom of revealed preference)} \end{aligned}$$