EC4101

Topic 3: General Equilibrium

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1 Reading

1. Snyder and Nicholoson, Chapter 13, Microeconomic Theory: Basic Principles and Extensions, 11th edition, 2012
2. Jehle and Reny, Chapter 5, Advanced Microeconomic Theory, 3rd edition, 2011

2 Two-good Simple Model

1. Assumptions:
   (a) Two consumer goods: x and y
   (b) Two factors of production: labor and capital.
   (c) All consumers have the same preference.
   (d) Consumers are price-taker and maximize utility (subject to budget constraint). Note that budget is endogenously determined.
   (e) All producers have the same production technology
   (f) Producers are price-taker and maximize profit.
2. Simple Graphical Analysis:
   (a) **Edgeworth box:** can be drawn to show efficient allocations.

(b) **Production possibility frontier (PPF)**: visualization of tradeoff between production of two consumer goods. Derived from edgeworth box.The slope is **rate of product transformation (RPT)**.

$$\begin{aligned} RPT_{x,y} &= -\left.\frac{dy}{dx}\right|_{\text{along PPF}} \\ &= -\left.\frac{dy}{dx}\right|_{C(x,y)=\bar{C}} = -\frac{C_x}{C_y} = -\frac{MC_x}{MC_y} \end{aligned}$$

(c) **Equilibrium prices and allocation**: when PPF is tangent to indifference curve. The slope will be relative prices of consumer goods. The relative prices of inputs (labor and capital) can be deterined in the edgeworth box with labor and capital. See Fig. 13.4 in the textbook. Comparative staics are be done on the graph too.

3 Model of Exchange

No production.
There are n goods and m consumers.
Goods are characterized by time, location and state of the world.
Consumers are endowed with goods to trade with each other.
Consumers are utility-maximizing price-takers (different consumers may have difference preferences.)
Consumption bundle of consumer i: $x^i = (x^i_1, x^i_2, \dots, x^i_n)$ where x^i_k is the consumption on good k.
Utility of consumer i: $u^i(x^i) = u^i(x^i_1, x^i_2, \dots, x^i_n)$
Endowment: $\bar{x}^i = (\bar{x}^i_1, \bar{x}^i_2, \dots, \bar{x}^i_n)$
Price of good k: p_k
Price vector: $p = (p_1, p_2, \dots, p_n)$
Consumers are price-taker: budget constraint is then

$$p_1x^i_1 + p_2x^i_2 + \cdots + p_nx^i_n \leq p_1\bar{x}^i_1 + p_2\bar{x}^i_2 + \cdots + p_n\bar{x}^i_n$$

or in vector form

$$px^i \leq p\bar{x}^i$$

Feasible allocation: $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^{nm}_{+}$ such that $\sum_{i=1}^{m} x^i \leq \sum_{i=1}^{m} \bar{x}^i$
Walrasian/Competitive/Market equilibrium: price vector $p \in \mathbb{R}^n_+$ and feasible allocation x such that
(i) Budget constraints: $px^i \leq p\bar{x}^i$ and
(ii) Utility maximization: for all $i = 1, \ldots, n$

$$u^i(x^i) \geq u^i(\tilde{x}^i)$$

for all $\tilde{x}^i$ such that $p\tilde{x}^i \leq p\bar{x}^i$
From Topic 1, if preference is continuous, strong monotonic, strictly quasi-concave, we have Marshallian demand functions:

$$x^i(p, p\bar{x}^i) = (x^i_1(p, p\bar{x}^i), x^i_2(p, p\bar{x}^i), \ldots, x^i_n(p, p\bar{x}^i)).$$

Excess demand: $z(p) = (z_1(p), z_2(p), \ldots, z_n(p))$ where

$$z_k(p) = \sum_{i=1}^{m} x^i_k(p, p\bar{x}^i) - \sum_{i=1}^{m} \bar{x}^i_k$$

Walrasian equilibrium price is a price vector p such that $z_k(p) \leq 0$ for all $k = 1, \ldots, m$.
Hence, if prices are strictly positive, then a Walrasian equilibrium is a pair (p, x) such that $x^i = x^i(p, p\bar{x}^i)$ for all $i = 1, \ldots, n$ and

$$\sum_{i=1}^{m} x^i(p, p\bar{x}^i) = \sum_{i=1}^{m} \bar{x}^i.$$

It is homogeneous of degree zero in all prices and income.

$$x^i(tp, tp\bar{x}^i) = x^i(p, p\bar{x}^i)$$

for any $t > 0$. This implies we can normalize prices. Holding relative prices fixed is sufficient.
Proposition. If preference is continuous, strong monotonic, strictly quasi-concave, we have Marshallian demand functions, we have
1. Continuity: $z(p)$ is continuous function of p (Berge's maximum Theorem)
2. Homogeneity: $z(p)$ is homogenous of degree zero in p: $z(\lambda p) = z(p)$ for all $\lambda > 0$.

(Follows from homogeneity of marshallian demand)
3. **Walras law**: $pz(p) = 0$ (budget constraints are binding under strong monotonicity; summing all inequalities over all individuals; reverse the order of summation.)
Walras law: value of excess demand is zero. When $n-1$ markets are clear in equilibrium, the last market will also be clear.
Since prices can be normalized, we adopt the usual normalization: $\sum_{i=1}^{n} p_i = 1$. Hence, prices is in a $n-1$ simplex $\Delta^n \equiv \{p \in \mathbb{R}^n : \sum_{i=1}^{n} p_i = 1 \text{ and } 0 \leq p_i \leq 1 \text{ for all } i = 1, \ldots, n\}$.
Existence proof of Walrasian equilibrium
If consumers' utility is continuous, strong monotonic, strictly quasi-concave, and all endowment are positive $\sum_{i=1}^{m} \bar{x}_k^i > 0$ for all $k = 1, \ldots, n$, then there exists a price vector p^* such that $z(p) \leq 0$.
Proof.
First, note that $z(p)$ is mapping from Δ^{n-1} to $\mathbb{R}^n$.
Define $g : \Delta^{n-1} \to \Delta^{n-1}$ by

$$g_k(p) = \frac{p_k + \max(0, z_k(p))}{1 + \sum_{\ell=1}^{n} \max(0, z_\ell(p))}, \text{ for } k = 1, ..., n.$$

Note that the $g_k(p)$ map is continuous since $z(p)$ and the max function are continuous functions.
Furthermore, $g(p) = (g_1(p), \ldots, g_n(p))$ is a point in the simplex Δ^{n-1} since $\sum_{k=1}^{n} g_k(p) = 1$ and $0 \leq g_k(p) \leq 1$ for all $k = 1, ..., n$.
This g also has a reasonable economic interpretation: if there is excess demand in some market, so that $z_k(p) \geq 0$, then the relative price of that good is increased.
By Brouwer's fixed point theorem (every continuous function g from a convex compact subset K of a Euclidean space to K itself has a fixed point.), there exists a p^* such that $p^* = g(p^*)$, i.e.,

$$p_k^* = \frac{p_k^* + \max(0, z_k(p^*))}{1 + \sum_{\ell=1}^{n} \max(0, z_\ell(p^*))}, \text{ for } k = 1, ..., n.$$

Now what remains to show it that p^* is a Walrasian equilibrium price.

Cross-multiply the equation and rearrange to obtain:

$$p_k^* \sum_{\ell=1}^{n} \max(0, z_\ell(p^*)) = \max(0, z_k(p^*)), \qquad k = 1, ...n.$$

Now multiply each of these n equations by $z_k(p^*)$:

$$z_k(p^*) p_k^* \sum_{\ell=1}^{n} \max(0, z_\ell(p^*)) = z_k(p^*) \max(0, z_k(p^*)), \qquad k = 1, ..., n.$$

Sum these n equations to obtain:

$$\sum_{k=1}^{n} p_k^* z_k(p^*) \sum_{\ell=1}^{n} \max(0, z_\ell(p^*)) = \sum_{k=1}^{n} z_k(p^*) \max(0, z_k(p^*)).$$

Now since $\sum_{k=1}^{n} p_k^* z_k(p^*) = 0$ by Walras' law, we have

$$\sum_{k=1}^{n} z_k(p^*) \max(0, z_k(p^*)) = 0.$$

Each term of this sum is greater than or equal to zero since each term is either 0 or $(z_k(p^*))^2$.
Hence, we have

$$z_k(p^*) \leq 0, \qquad k = 1, ..., n.$$

We have thus completed the equilibrium existence argument. **Q.E.D**

Pareto efficient allocation.
First welfare theorem. If preference is local-nonsatisable (for any bundle of goods there is always another bundle of goods arbitrarily close that is preferred to it), a Walrasian equlibrium allocation is Pareto efficient.
Proof.
Suppose the contrary that (p, x) is Walrasian equilibrium but x is not Pareto efficient. Hence, there is a feasible allocation $\hat{x}$ that $u_i(\hat{x}) \geq u_i(x)$ for all $i = 1, \ldots, m$ and $u_i(\hat{x}) > u_i(x)$ for some $i = 1, \ldots, m$.

Suppose $p\hat{x}^i < px^i$ for some $i = 1, \ldots, m$. Then by local-nonsatisation, then consumer i can choose $\tilde{x}_i$ under prices p which is strictly better. Hence, $p\hat{x}^i \geq px^i$ for all $i = 1, \ldots, m$.
Suppose $p\hat{x}^i = px^i$ for all $i = 1, \ldots, m$. Then consumer i can choose $\hat{x}_i$ under prices p but it is is strictly better. Hence,

$$\begin{aligned} p\hat{x}^i &\geq px^i \text{ for all } i = 1, \ldots, n \\ p\hat{x}^i &> px^i \text{ for some } i = 1, \ldots, n \end{aligned}$$

and summing up implies

$$\sum_{i=1}^{m} p\hat{x}^i > \sum_{i=1}^{m} px^i$$

Under local non-satiation, budget constraint must hold (see Topic 1), hence, we have

$$\sum_{i=1}^{m} px^i = \sum_{i=1}^{m} p\bar{x}^i$$

Therefore,

$$\sum_{i=1}^{m} p\hat{x}^i > \sum_{i=1}^{m} p\bar{x}^i$$

However, by feasibility of $\hat{x}$, we have $\sum_{i=1}^{m} x_k^i \leq \sum_{i=1}^{m} \bar{x}_k^i$ for all $k = 1, \ldots, n$.
For any non-negative prices, we have

$$\sum_{i=1}^{m} p_k x_k^i \leq \sum_{i=1}^{m} p_k \bar{x}_k^i \text{ for all } k = 1, \ldots, n.$$

Then, summing all k equations, we have

$$\sum_{i=1}^{m} \sum_{k=1}^{n} p_k x_k^i \leq \sum_{i=1}^{m} \sum_{k=1}^{n} p_k \bar{x}_k^i$$

or compactly

$$\sum_{i=1}^{m} p\hat{x}^i \leq \sum_{i=1}^{m} p\bar{x}^i$$

which is a contradiction. QED

Second welfare theorem. Every Pareto efficient allocation can be obtained as a Walarsian equilibrium allocation by redistribution of endowment.
Proof. Omitted.

4 Extension

4.1 Core

Let $M = \{1, \dots, m\}$ be the set of consumers.
An allocation x is a core allocation if there exists no other allocation $\hat{x}$ such that for some group of consumers $S \subseteq M$ such that $u_i(\hat{x}^i) \geq u_i(x^i)$ for all $i \in S$ and $u_i(\hat{x}^i) > u_i(x^i)$ for some $i \in S$ where $\sum_{i \in S} x^i \leq \sum_{i \in S} \bar{x}^i$.
All core allocation are pareto efficient but not a Pareto efficient allocation might not be a core allocation.
Core and competitive equilibria: Competitive equilibria are inside the core.
r-**replica economy**: replicate the economy r times that means we have r consumer 1 with the same level of endowment.
Edgeworth-Debreu-Scarf Limit Theorem on Core. Competitive equilibrium in an ∞-replica economy is the core
Proof. Omitted.

4.2 Contingent plan

General equilibirum so far does not consider goods with uncertainty. It can be easily incorporate by consider goods in different states are in fact separate goods.

5 Production

There are n goods, m consumers and f firms.
Goods are characterized by time, location and state of the world.
Consumers are endowed with goods to trade with each other.
Consumers are utility-maximizing price-takers (different consumers may have difference preferences.)
Consumption bundle of consumer i: $x^i = (x_1^i, x_2^i, \dots, x_n^i)$ where x_k^i is the consumption on good k.

Utility of consumer i: $u^i(x^i) = u^i(x_1^i, x_2^i, \dots, x_n^i)$
Endowment: $\bar{x}^i = (\bar{x}_1^i, \bar{x}_2^i, \dots, \bar{x}_n^i)$
Price of good k: p_k
Price vector: $p = (p_1, p_2, \dots, p_n)$
Consumer i owns a share $\theta_{ij} \geq 0$ of the j-th firm.
Production plan of firm j: $y^j = (y_1^i, y_2^i, \dots, y_n^i)$ where y_k^i is the net output of good k. (Hence, input is negative and output is positive) Assume that technology set is convex.
Production plan $y = (y^1, \dots, y^f)$
Budget constraint becomes

$$px^i \leq \sum\nolimits_j \theta_{ij} p y^j + p\bar{x}^i$$

The excess demand:

$$z(p) = \sum_i x^i - \sum\nolimits_j y_j - \sum\nolimits_i \bar{x}^i$$

Walrasian/Competitive/Market equilibrium: price vector $p \in \mathbb{R}_+^n$, production plan y and feasible allocation x such that
(i) Budget constraints: $px^i \leq \sum_j \theta_{ij} p y^j + p\bar{x}^i$ and
(ii) Utility maximization: for all $i = 1, \dots, n$

$$u^i(x^i) \geq u^i(\tilde{x}^i)$$

for all $\tilde{x}^i$ such that $p\tilde{x}^i \leq \sum_j \theta_{ij} p y^j + p\bar{x}^i$
(iii) Profit maximization: for all $j = 1, \dots, f$

$$py^j \geq p\tilde{y}^j$$

for all feasible $\tilde{y}^j$.
Two welfare theorems continues to hold. Formal statements and proofs are omitted.

A Appendix.

1. Standard n-simplex (or unit n-simplex) is a subset of $\mathbb{R}^{n+1}$ given by

$$\Delta^n = \{(t_0, t_1, \dots, t_n) : \sum\nolimits_{i=1}^n t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i\}$$

2. **Brouwer Fixed Point Theorem**: If $f : \Delta^{n-1} \to \Delta^{n-1}$ is a continuous function,

then there exists a fixed point in Δ^{n-1}: there is a $x \in \Delta^{n-1}$ that $x = f(x)$.