EC4101
Microeconomics Analysis III
(Group 2)

Topic 2

Theory of Firm

What are we going to do today?

- Production Function
  - Marginal product, Isoquant, Marginal rate of Technical Substitution, Return to Scale, Elasticity of substitution
- Cost Function
  - Cost minimization, firm's expansion path, total cost function, average cost, marginal cost, input substitution, contingent demand for inputs, short-run and long-run
- Profit maximization
  - Marginal revenue, profit maximization, elasticity and mark up, short-run supply of price-taking firm, profit function, producer surplus, input demand function, cross price effect (substitution + output effect)

Production Function

Production function

- Quantity (q) based on captial (k) and labor (l)

$$q = f(k,l)$$

- In general, quantity can depends on n different factors ($r_1,r_2,\ldots,r_n$)

$$q = f(r_1,r_2,\ldots,r_n)$$

Marginal product

- Definition:

$$\text{marginal physical product of capital} = MP_k = \frac{\partial q}{\partial k} = f_k$$

$$\text{marginal physical product of labor} = MP_l = \frac{\partial q}{\partial l} = f_l$$

- Standard assumption: diminishing

$$\frac{\partial MP_k}{\partial k} = \frac{\partial^2 f}{\partial k^2} = f_{kk} = f_{11} < 0$$

$$\frac{\partial MP_l}{\partial l} = \frac{\partial^2 f}{\partial l^2} = f_{ll} = f_{22} < 0$$

Average Product

- Definition

$$AP_l = \frac{\text{output}}{\text{labor input}} = \frac{q}{l} = \frac{f(k,l)}{l}$$

EXAMPLE 9.1 A Two-Input Production Function

- Suppose the production function for flyswatters can be represented by

$$q = f(k,l) = 600k^2l^2 - k^3l^3$$

- To construct MP_l and AP_l, we must assume a value for k
  - Let $k = 10$
- The production function becomes

$$q = 60{,}000l^2 - 1000l^3$$

EXAMPLE 9.1 A Two-Input Production Function

- The marginal productivity function is

$$MP_l = \partial q/\partial l = 120{,}000l - 3000l^2$$

  - Which diminishes as l increases
  - This implies that q has a maximum value:

$$120{,}000l - 3000l^2 = 0$$

$$40l = l^2$$

$$l = 40$$

- Labor input beyond $l = 40$ reduces output

EXAMPLE 9.1 A Two-Input Production Function

- To find average productivity, we hold k=10 and solve

$$AP_l = q/l = 60{,}000l - 1000l^2$$

- AP_l reaches its maximum where

$$\partial AP_l/\partial l = 60{,}000 - 2000l = 0$$

$$l = 30$$

- When $l = 30$, $AP_l = MP_l = 900{,}000$
  - When AP_l is at its maximum, AP_l and MP_l are equal

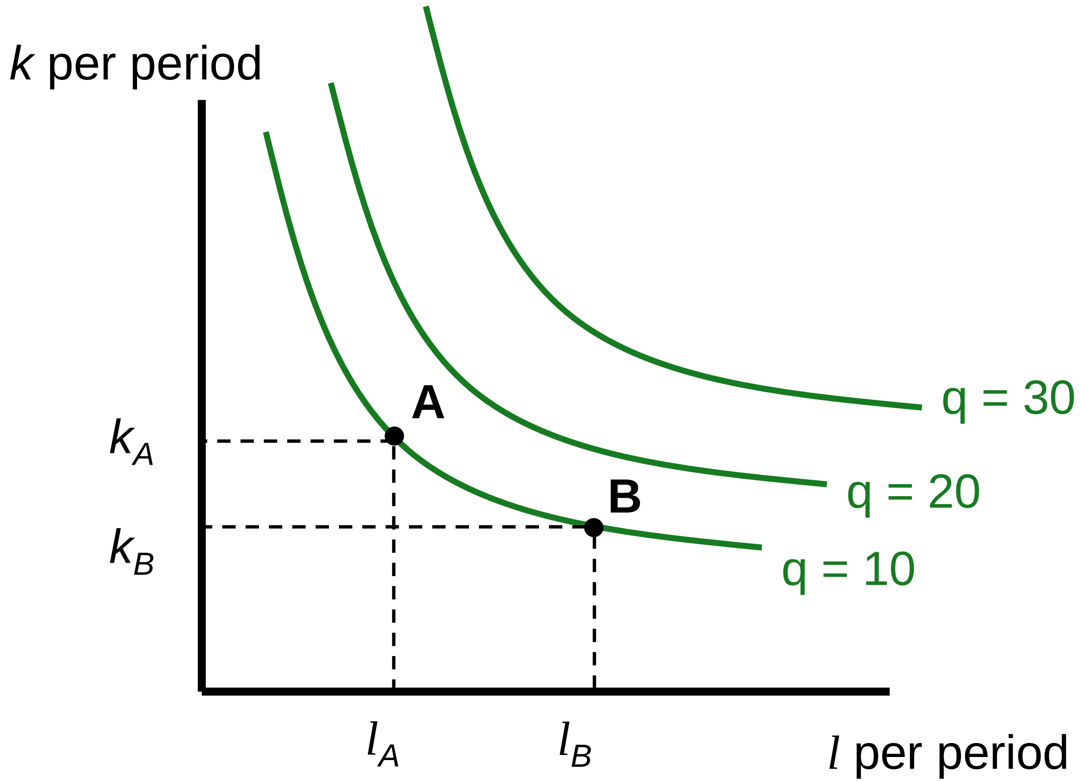
Isoquant

- Graphical representation of production function
- Slope is Marginal rate of technical substitution RTS

$$RTS\ (l \text{ for } k) = \left.\frac{-dk}{dl}\right|_{q=q_0}$$

FIGURE 9.1

An Isoquant Map



Isoquants record the alternative combinations of inputs that can be used to produce a given level of output. The slope of these curves shows the rate at which l can be substituted for k while keeping output constant. The negative of this slope is called the (marginal) rate of technical substitution (RTS). In the figure, the RTS is positive and diminishing for increasing inputs of labor.

RTS and Marginal Productivities

- Total differential of the production function:

$$dq = \frac{\partial f}{\partial l} \cdot dl + \frac{\partial f}{\partial k} \cdot dk = MP_l \cdot dl + MP_k \cdot dk$$

- Along an isoquant dq = 0, so

$$MP_l \cdot dl = -MP_k \cdot dk$$

$$RTS \ (l \text{ for } k) = \left. \frac{-dk}{dl} \right|_{q=q_0} = \frac{MP_l}{MP_k}$$

Property of RTS

- Always positive (why?)
- Diminishing return $\nRightarrow$ Diminishing RTS
  - (why? Similar to consumer theory)

Return to Scale

- Suppose that all inputs are doubled, would output double?

Effect on Output	Returns to Scale
$f(tk,tl) = tf(k,l) = tq$	Constant
$f(tk,tl) < tf(k,l) = tq$	Decreasing
$f(tk,tl) > tf(k,l) = tq$	Increasing

- Not related to RTS

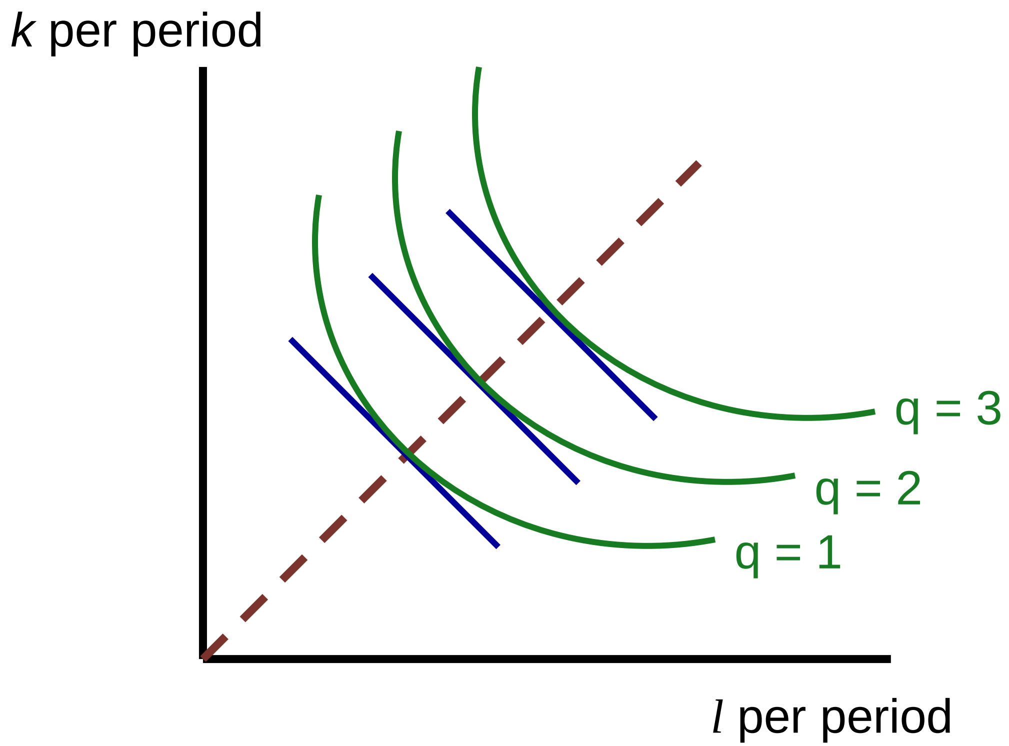
Constant returns-to-scale

- production functions: homogeneous of degree one in inputs

$$f(tk,tl) = t^1 f(k,l) = tq$$

- MP_L and MP_K: homogeneous of degree zero
- Isoquants: radial expansions

Isoquant Map for a Constant Returns-to-Scale Production Function



Because a constant returns-to-scale production function is homothetic, the RTS depends only on the ratio of *k* to *l*, not on the scale of production. Consequently, along any ray through the origin (a ray of constant *k/l*), the RTS will be the same on all isoquants. An additional feature is that the isoquant labels increase proportionately with the inputs.

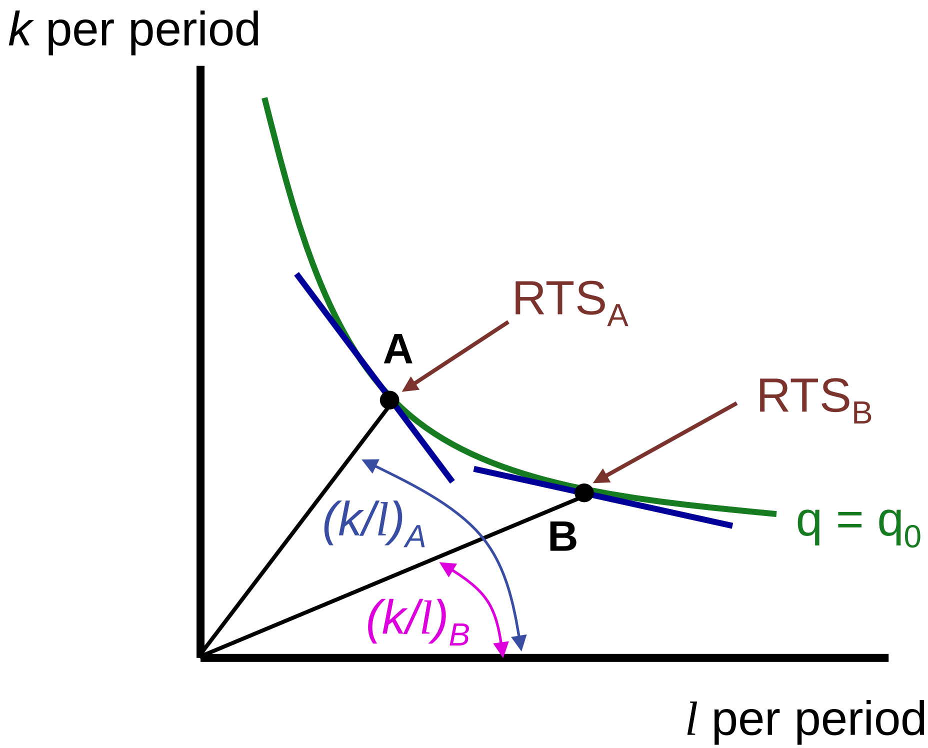
Curvature of Isoquant

- Elasticity of substitution (σ)
  - Measures the $\%\Delta k/l$ relative to the proportionate $\%\Delta$ *RTS* along an isoquant

$$\sigma = \frac{\%\Delta(k/l)}{\%\Delta RTS} = \frac{d(k/l)}{dRTS} \times \frac{RTS}{k/l} = \frac{d\ln(k/l)}{d\ln RTS} = \frac{d\ln(k/l)}{d\ln(f_1/f_k)}$$

  - always be positive: *k/l* and *RTS* move in the same direction
  - σ is high: isoquant will be relatively flat
  - σ is low: isoquant will be sharply curved

Graphic Description of the Elasticity of Substitution



In moving from point A to point B on the $q = q_0$ isoquant, both the capital–labor ratio (k/l) and the RTS will change. The elasticity of substitution (σ) is defined to be the ratio of these proportional changes; it is a measure of how curved the isoquant is.

Standard Production Function

- Linear Production
- Fixed Proportion
- Cobb-Douglas
- CES

The Linear Production Function

- Linear production function ($\sigma = \infty$):

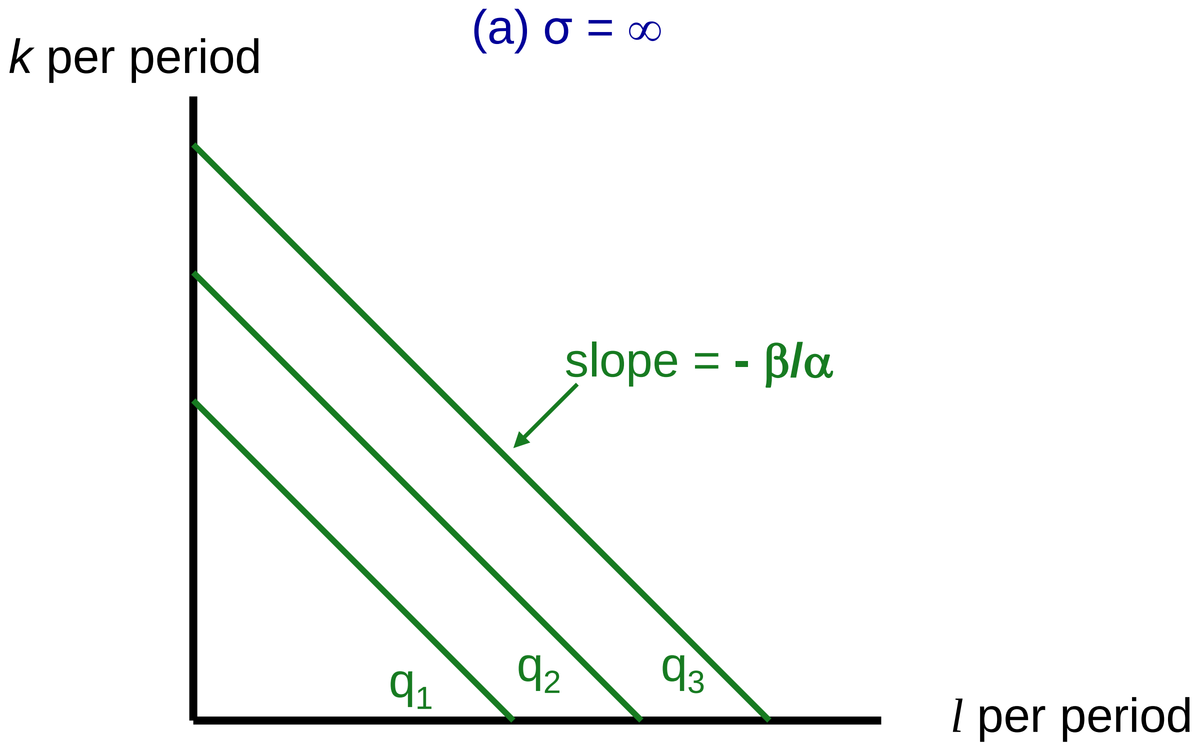
$$q = f(k,l) = \alpha k + \beta l$$

  – Constant returns to scale

$$f(tk,tl) = \alpha\, tk + \beta\, tl = t(\alpha\, k + \beta\, l) = tf(k,l)$$

  – All isoquants are straight lines with slope -α/β
    - *RTS* is constant
    - $\sigma = \infty$

Isoquant Maps for Simple Production Functions with Various Values for σ



Three possible values for the elasticity of substitution are illustrated in these figures. In (a), capital and labor are perfect substitutes. In this case, the RTS will not change as the capital–labor ratio changes.

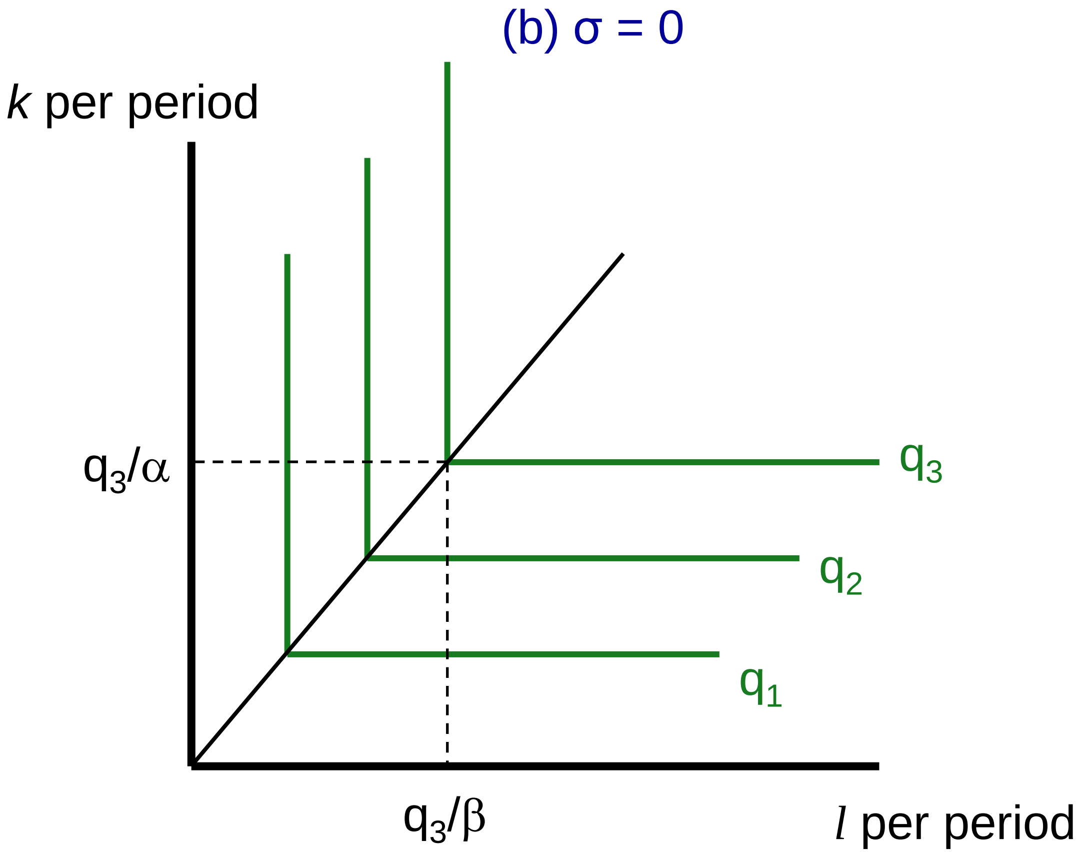
Fixed Proportions

- Fixed proportions production function (σ = 0):

$$q = \min(\alpha k, \beta l) \quad \alpha, \beta > 0$$

  - Capital and labor must always be used in a fixed ratio
    - The firm will always operate along a ray where k/l is constant
  - Because k/l is constant, $\sigma = 0$

Isoquant Maps for Simple Production Functions with Various Values for σ



Three possible values for the elasticity of substitution are illustrated in these figures.

In (b), the fixed–proportions case, no substitution is possible. The capital–labor ratio is fixed at β/α.

Cobb-Douglas Production Function

- Cobb-Douglas production function ($\sigma = 1$):

$$q = f(k,l) = Ak^{\alpha}l^{\beta} \quad A,\alpha,\beta > 0$$

- This production function can exhibit any returns to scale

$$f(tk,tl) = A(tk)^{\alpha}(tl)^{\beta} = At^{\alpha+\beta}k^{\alpha}l^{\beta} = t^{\alpha+\beta}f(k,l)$$

  – if $\alpha + \beta = 1 \Rightarrow$ constant returns to scale
  – if $\alpha + \beta > 1 \Rightarrow$ increasing returns to scale
  – if $\alpha + \beta < 1 \Rightarrow$ decreasing returns to scale

Cobb-Douglas Production Function

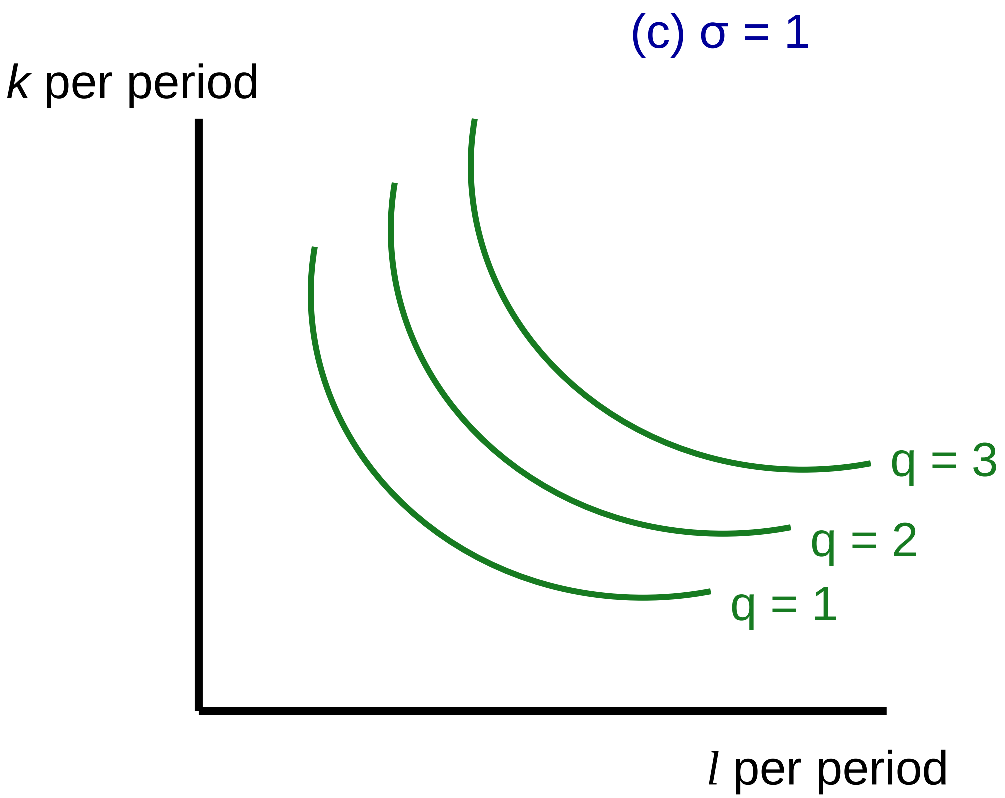
- The Cobb-Douglas production function is linear in logarithms:

$$\ln q = \ln A + \alpha \ln k + \beta \ln l$$

  - α is the elasticity of output with respect to k
  - β is the elasticity of output with respect to l

FIGURE 9.4 (c)

Isoquant Maps for Simple Production Functions with Various Values for σ



Three possible values for the elasticity of substitution are illustrated in these figures.
A case of limited substitutability is illustrated in (c).

CES Production Function

- CES production function ($\sigma = 1/(1-\rho)$):

  $q = f(k,l) = [k^{\rho} + l^{\rho}]^{\gamma/\rho}$ $\rho \leq 1, \rho \neq 0, \gamma > 0$

  - $\gamma > 1 \Rightarrow$ increasing returns to scale
  - $\gamma < 1 \Rightarrow$ decreasing returns to scale
  - For this production function, $\sigma = 1/(1-\rho)$
    - $\rho = 1 \Rightarrow$ linear production function
    - $\rho = -\infty \Rightarrow$ fixed proportions production function
    - $\rho = 0 \Rightarrow$ Cobb-Douglas production function

EXAMPLE 9.3 A Generalized Leontief Production Function

- Production function: $q = f(k,l) = k + l + 2(kl)^{0.5}$
  - Constant returns to scale
  - Marginal productivities are

    $f_k = 1 + (k/l)^{-0.5}$ and $f_l = 1 + (k/l)^{0.5}$
  - RTS diminishes as k/l falls

$$RTS = \frac{f_l}{f_k} = \frac{1+(k/l)^{0.5}}{1+(k/l)^{-0.5}}$$

  - This function has a CES form ($\rho = 0.5$ and $\gamma = 1$)
  - Elasticity of substitution:

$$\sigma = \frac{1}{1-\rho} = \frac{1}{0.5} = 2$$

Many-input production functions

- Many-input Cobb–Douglas: $q = \prod_{i=1}^{n} x_i^{\alpha_i}$
  - Constant returns to scale if $\sum_{i=1}^{n} \alpha_i = 1$
    - α_i is the elasticity of q with respect to input x_i.
    - Because $0 < \alpha_i < 1$, each input exhibits diminishing marginal productivity
  - Any degree of increasing returns to scale can be incorporated, depending on $\varepsilon = \sum_{i=1}^{n} \alpha_i$
  - The elasticity of substitution between any two inputs is 1

Many-input production functions

- Many-input constant elasticity of substitution (CES):

$$q = \left[\sum \alpha_i x_i^{\rho} \right]^{\varepsilon/\rho}, \quad \rho \le 1$$

  - Constant returns to scale for ε=1
  - Diminishing marginal productivities for each input because ρ ≤ 1
  - The elasticity of substitution: σ=1/(1-ρ)

Many-input production functions

- Nested production functions
  - Cobb–Douglas and CES production functions are combined into a “nested” single function

Composite input x_4 (CES):

$$x_4 = \left[\gamma x_1^{\rho} + (1-\gamma) x_2^{\rho}\right]^{1/\rho}$$

Final production function (Cobb – Douglas):

$$q = x_3^{\alpha} x_4^{\beta}$$

Many-input production functions

- Generalized Leontief:

$$q = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \sqrt{x_i x_j}, \quad \text{where} \quad \alpha_{ij} = \alpha_{ji}$$

  - Constant returns to scale
  - Diminishing marginal productivities to all inputs
    - Because each input appears both linearly and under the radical
  - Symmetry of the second-order partial derivatives

Many-input production functions

- Translog:

$$\ln q = \alpha_0 + \sum_{i=1}^{n} \alpha_i \ln x_i + 0.5 \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \ln x_i \ln x_j, \quad \text{where} \quad \alpha_{ij} = \alpha_{ji}$$

  – Cobb-Douglas for $\alpha_0 = \alpha_{ij} = 0$ for all i,j

  – May assume any degree of returns to scale

  – The condition $\alpha_{ij} = \alpha_{ji}$ is required to ensure equality of the cross-partial derivatives

Cost Functions

Cost

- Price-taker in the factor market
  - Total costs: $C = wl + vk$
- Cost Minimizing problem
  - Interior solution: $RTS = w/v$
  - rate at which k can be traded for l in the production process = the rate at which they can be traded in the marketplace

Cost-Minimizing Input Choices

- Minimize total costs given $q = f(k,l) = q_0$
- Setting up the Lagrangian:

$$\mathscr{L} = wl + vk + \lambda[q_0 - f(k,l)]$$

- First-order conditions:

$$\partial\mathscr{L}/\partial l = w - \lambda(\partial f/\partial l) = 0$$

$$\partial\mathscr{L}/\partial k = v - \lambda(\partial f/\partial k) = 0$$

$$\partial\mathscr{L}/\partial\lambda = q_0 - f(k,l) = 0$$

First Order conditions

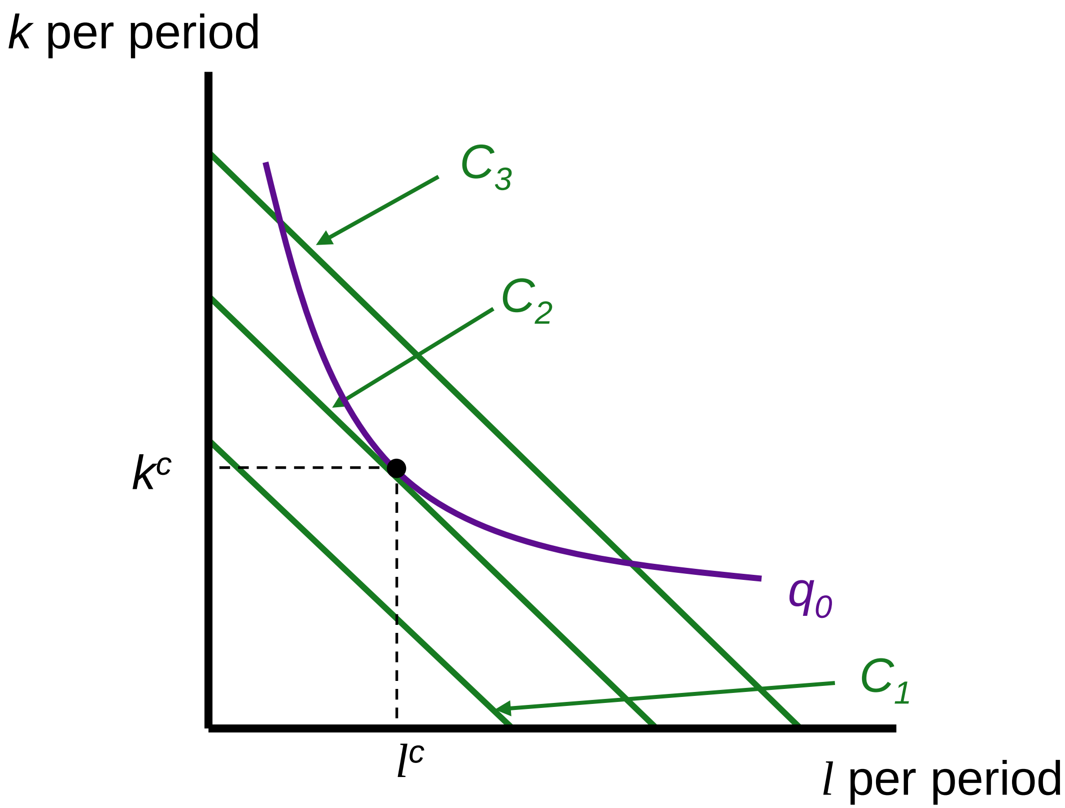
- Dividing the first two conditions we get

$$\frac{w}{v} = \frac{\partial f / \partial l}{\partial f / \partial k} = RTS \ (l \text{ for } k)$$

  - RTS = prices ratio (Same as MRS= prices ratio)
- Cross-multiplying, we get $\dfrac{f_k}{v} = \dfrac{f_l}{w}$
  - the marginal productivity per dollar spent should be the same for all inputs
- The inverse: $\dfrac{w}{f_l} = \dfrac{v}{f_k} = \lambda$
  - Lagrangian multiplier:how the extra costs that would be incurred by increasing the output constraint slightly

FIGURE 10.1

Minimization of Costs Given $q = q_0$

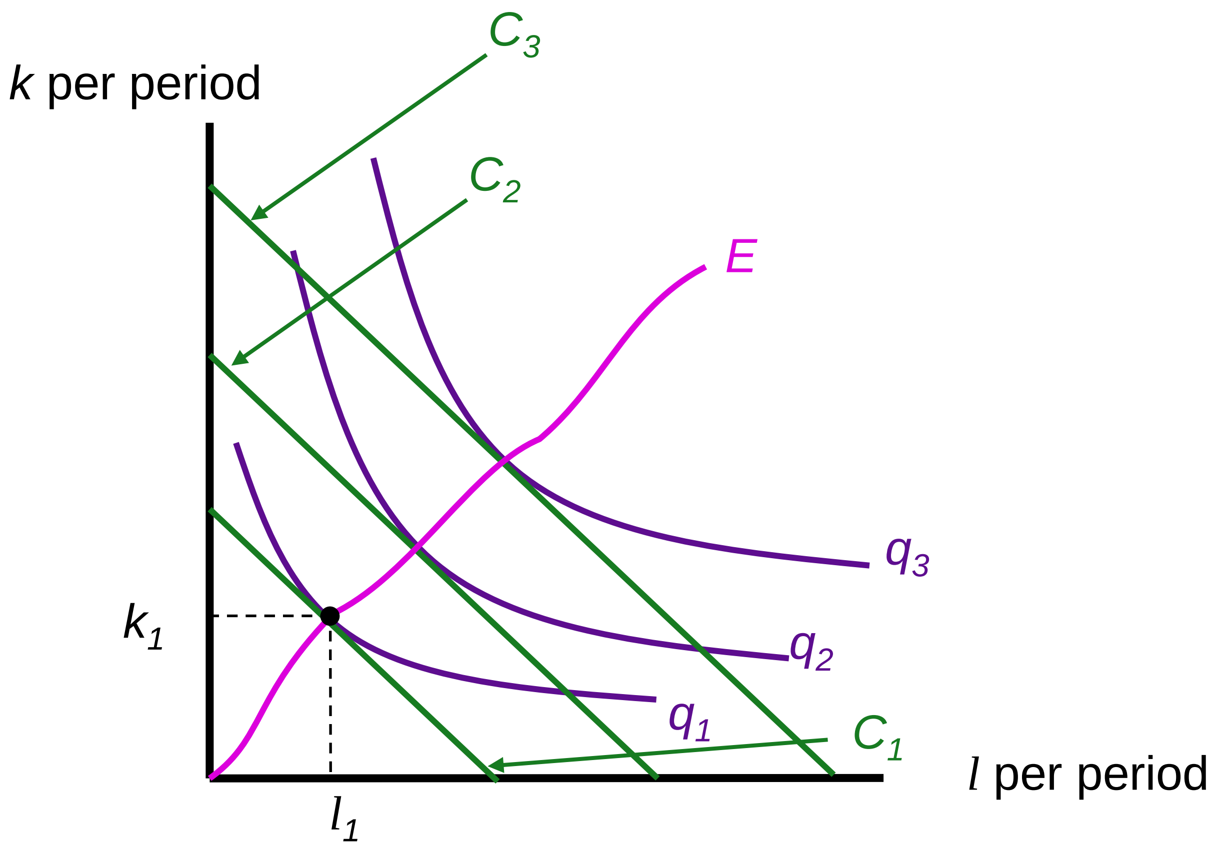


A firm is assumed to choose *k* and *l* to minimize total costs. The condition for this minimization is that the rate at which k and l can be traded technically (while keeping *q = q0*) should be equal to the rate at which these inputs can be traded in the market. In other words, the *RTS* (of *l for k*) should be set equal to the price ratio *w/v*. This tangency is shown in the figure; costs are minimized at C_1 by choosing inputs k^c and l^c.

Firm's expansion Path

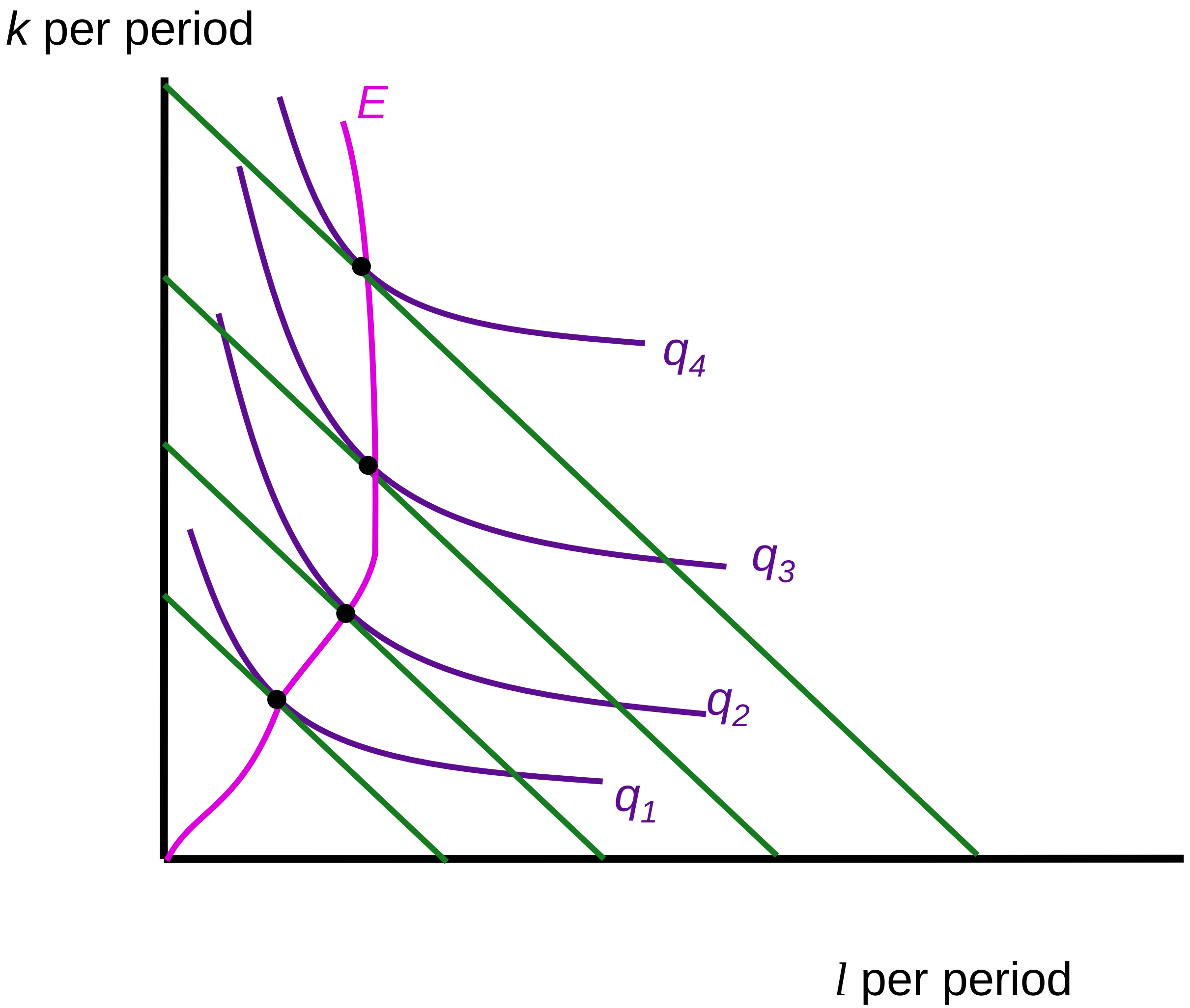
- Locus of cost-minimizing choices of k and l
- Can be downward sloping: if the use of an input falls as output expands, that input is an inferior input

The Firm's Expansion Path



The firm's expansion path is the locus of cost-minimizing tangencies. Assuming fixed input prices, the curve shows how inputs increase as output increases.

Input Inferiority



With this particular set of isoquants, labor is an inferior input because less l is chosen as output expands beyond q_2.

- Cobb-Douglas production function: $q = k^{\alpha} l^{\beta}$
  - The Lagrangian expression for cost minimization of producing q_0 is

$$\mathscr{L} = vk + wl + \lambda(q_0 - k^{\alpha} l^{\beta})$$

  - First-order conditions for a minimum

$$\partial\mathscr{L}/\partial k = v - \lambda\alpha k^{\alpha-1} l^{\beta} = 0$$

$$\partial\mathscr{L}/\partial l = w - \lambda\beta k^{\alpha} l^{\beta-1} = 0$$

$$\partial\mathscr{L}/\partial\lambda = q_0 - k^{\alpha} l^{\beta} = 0$$

- Dividing the first equation by the second gives us

$$\frac{w}{v} = \frac{\beta k^{\alpha} l^{\beta-1}}{\alpha k^{\alpha-1} l^{\beta}} = \frac{\beta}{\alpha} \cdot \frac{k}{l} = RTS$$

- This production function is homothetic
  - The RTS depends only on the ratio of the two inputs
  - The expansion path is a straight line

- CES production function: $q = (k^{\rho} + l^{\rho})^{\gamma/\rho}$
  - The Lagrangian expression for cost minimization of producing q_0 is

$$\mathscr{L} = vk + wl + \lambda[q_0 - (k^{\rho} + l^{\rho})^{\gamma/\rho}]$$

  - First-order conditions for a minimum

$$\partial\mathscr{L}/\partial k = v - \lambda(\gamma/\rho)(k^{\rho} + l^{\rho})^{(\gamma-\rho)/\rho}(\rho)k^{\rho-1} = 0$$

$$\partial\mathscr{L}/\partial l = w - \lambda(\gamma/\rho)(k^{\rho} + l^{\rho})^{(\gamma-\rho)/\rho}(\rho)l^{\rho-1} = 0$$

$$\partial\mathscr{L}/\partial\lambda = q_0 - (k^{\rho} + l^{\rho})^{\gamma/\rho} = 0$$

EXAMPLE 10.1 Cost Minimization

- Dividing the first equation by the second gives us

$$\frac{w}{v} = \left(\frac{1}{k}\right)^{\rho-1} = \left(\frac{k}{l}\right)^{1-\rho} = \left(\frac{k}{l}\right)^{1/\sigma}$$

- This production function is also homothetic

Total cost function

- Optimal-value function of cost minimization:

$$C = C(v,w,q)$$

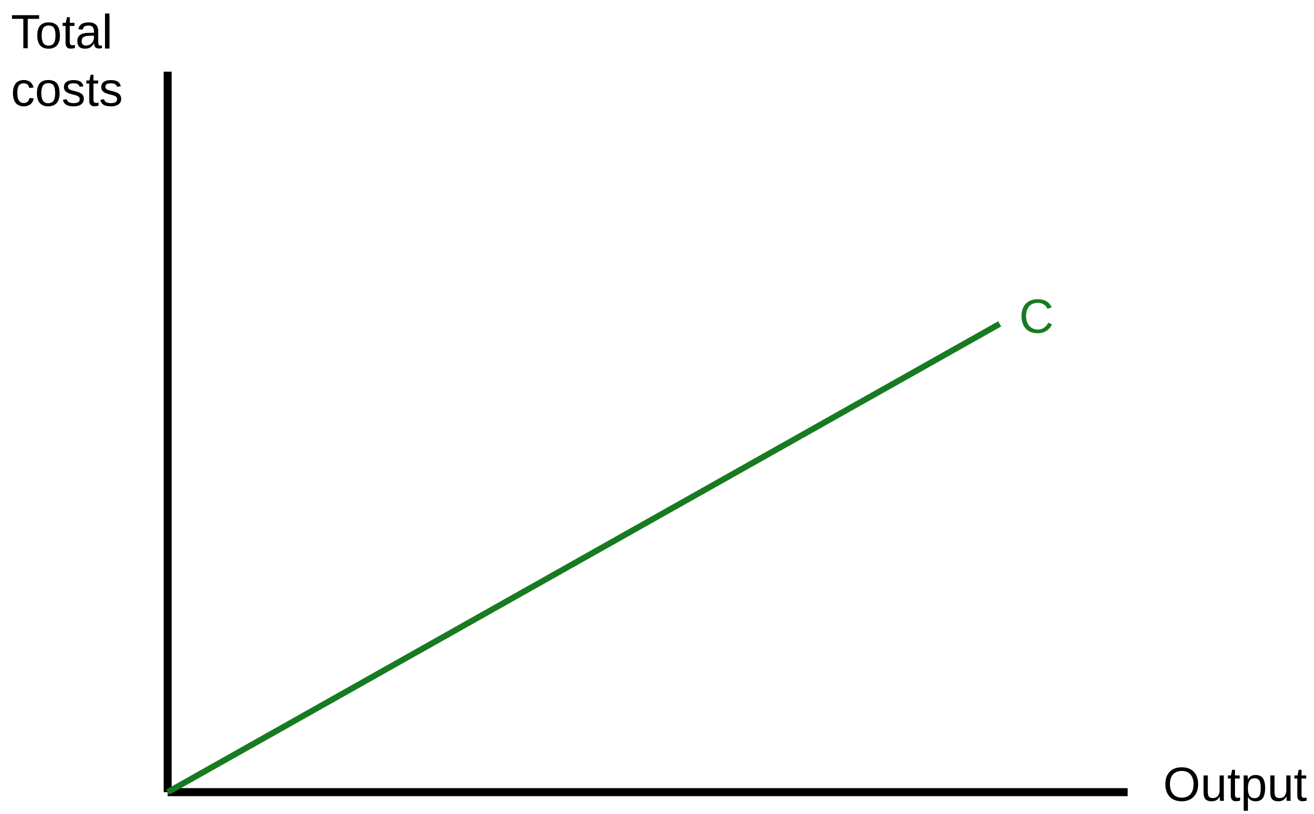
- Average cost (AC)

$$\text{average cost} \ = \ AC(v,w,q) = \frac{C(v,w,q)}{q}$$

- Marginal cost (MC)

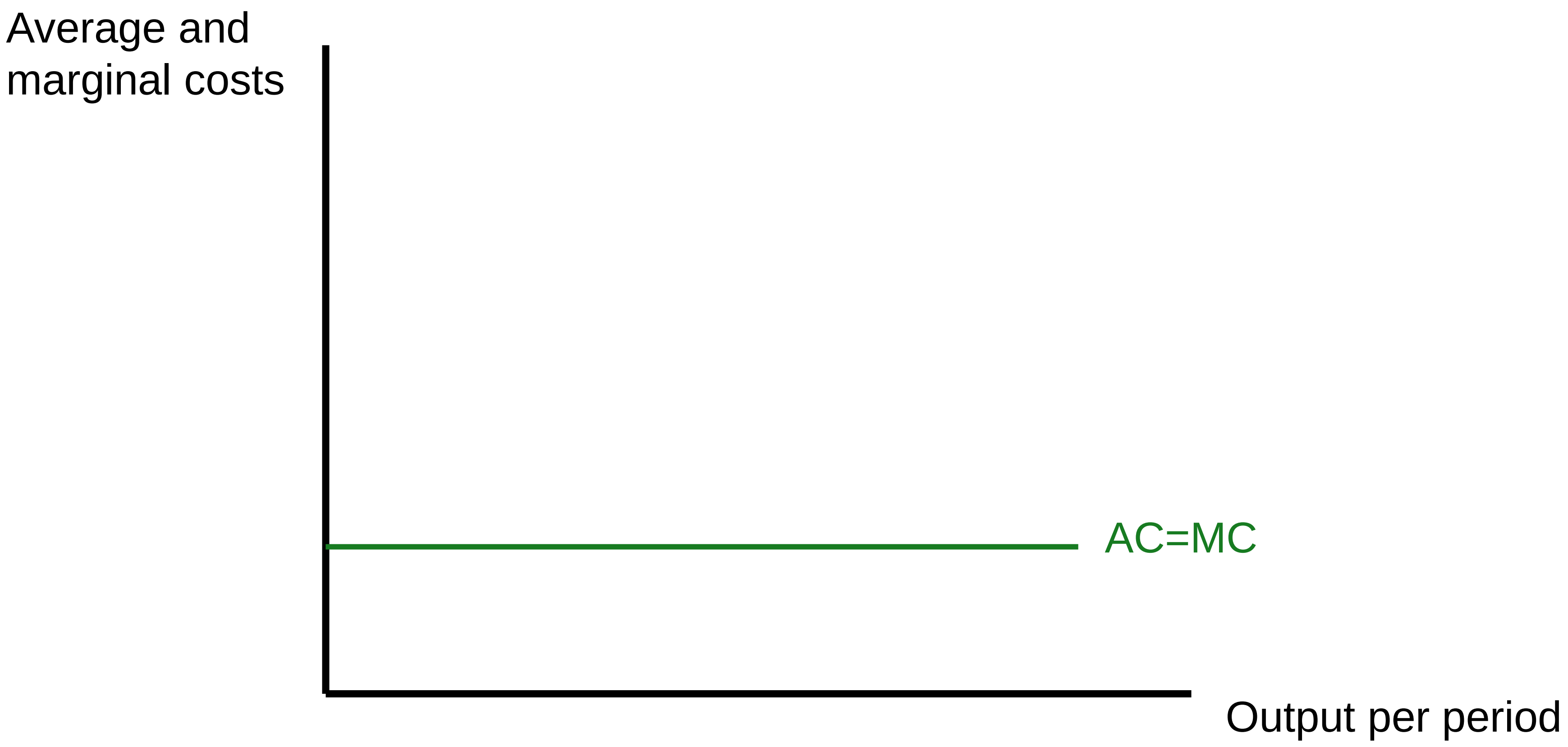
$$\text{marginal cost} \ = \ MC(v,w,q) = \frac{\partial C(v,w,q)}{\partial q}$$

Total, Average, and Marginal Cost Curves for the Constant Returns-to-Scale Case



In (a) total costs are proportional to output level.

Total, Average, and Marginal Cost Curves for the Constant Returns-to-Scale Case

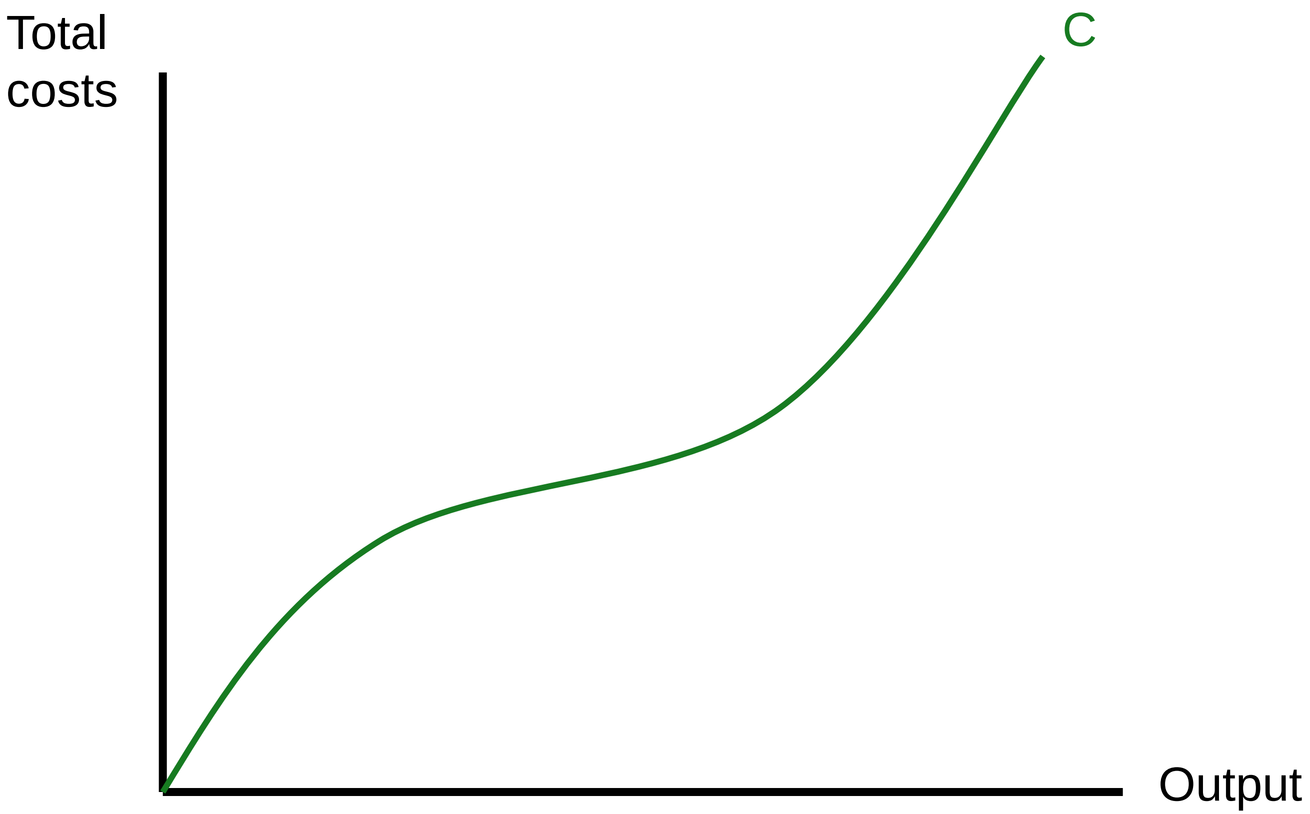


Average and marginal costs, as shown in (b), are equal and constant for all output levels.

U-Shaped AC

- Total costs start out as concave and then becomes convex as output increases

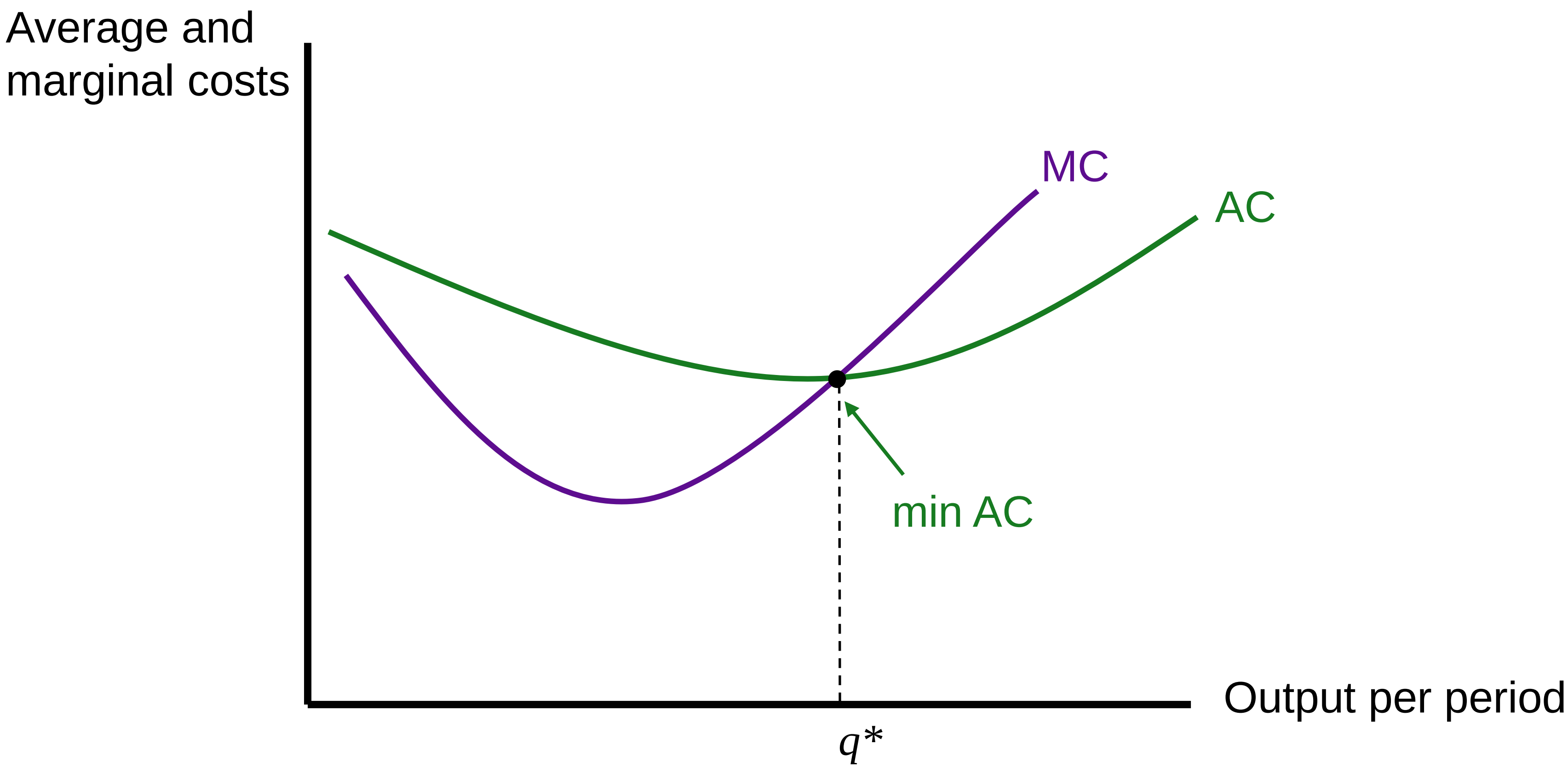
Total, Average, and Marginal Cost Curves for the Cubic Total Cost Curve Case



If the total cost curve has the cubic shape shown in (a), average and marginal cost curves will be U-shaped.

FIGURE 10.5 (b)

Total, Average, and Marginal Cost Curves for the Cubic Total Cost Curve Case



If the total cost curve has the cubic shape shown in (a), average and marginal cost curves will be U-shaped. In (b) the marginal cost curve passes through the low point of the average cost curve at output level q*.

- Fixed proportions

$$q = f(k,l) = \min(\alpha k, \beta l)$$

  - Production will occur at the vertex of the L-shaped isoquants ($q = ak = bl$)

$$C(w,v,q) = vk + wl = v(q/a) + w(q/b)$$

$$C(w,v,q) = a\left(\frac{v}{a} + \frac{w}{b}\right)$$

- Cobb-Douglas, $q = f(k,l) = k^{\alpha} l^{\beta}$
  - Cost minimization requires that:

$$\frac{w}{v} = \frac{\beta}{\alpha} \cdot \frac{k}{l}, \text{ so } \quad k = \frac{\alpha}{\beta} \cdot \frac{w}{v} \cdot l$$

  - Substitute into the production function and solve for *l*, then for *k*

$$l = q^{1/\alpha+\beta} \left(\frac{\beta}{\alpha} \right)^{\alpha/\alpha+\beta} w^{-\alpha/\alpha+\beta} v^{\alpha/\alpha+\beta}$$

$$k = q^{1/\alpha+\beta} \left(\frac{\alpha}{\beta} \right)^{\beta/\alpha+\beta} w^{\beta/\alpha+\beta} v^{-\beta/\alpha+\beta}$$

- Cobb-Douglas
  - Now we can derive total costs as

$$C(v,w,q) = vk + wl = q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{\beta/\alpha+\beta}$$

  - Where

$$B = (\alpha+\beta)\alpha^{-\alpha/\alpha+\beta} \beta^{-\beta/\alpha+\beta}$$

    - Which is a constant that involves only the parameters α and β

EXAMPLE 10.2 Some Illustrative Cost Functions

- CES: $q = f(k,l) = (k^{\rho} + l^{\rho})^{\gamma/\rho}$
  - To derive the total cost, we would use the same method and eventually get

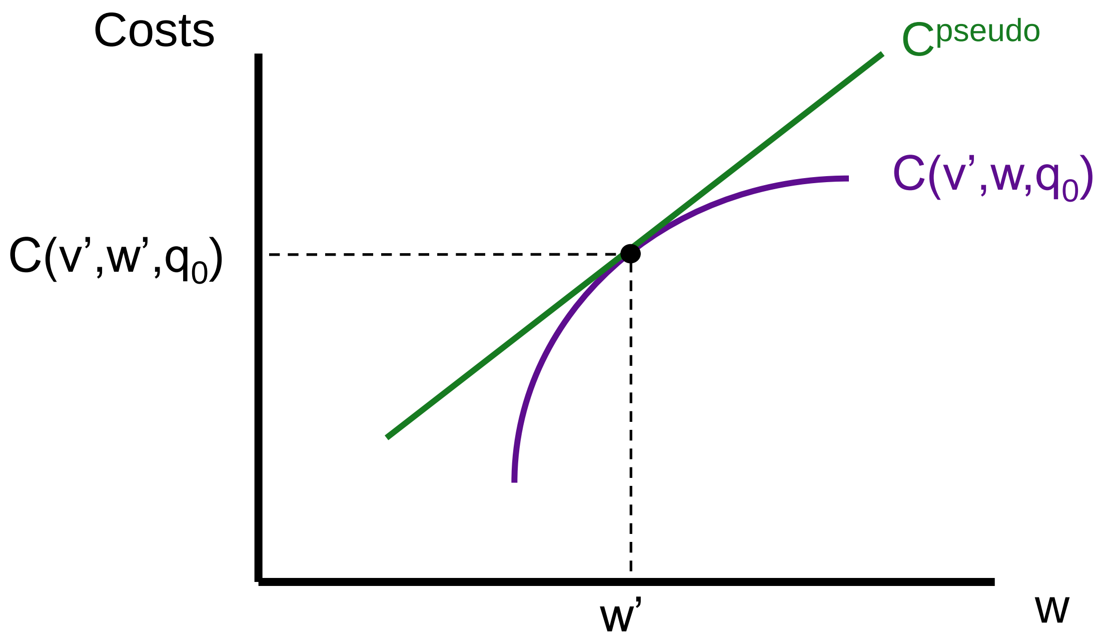
$$C(v,w,q) = vk + wl = q^{1/\gamma}(v^{\rho/\rho-1} + w^{\rho/\rho-1})^{(\rho-1)/\rho}$$

$$C(v,w,q) = q^{1/\gamma}(v^{1-\sigma} + w^{1-\sigma})^{1/1-\sigma}$$

Properties of Cost Functions

- Homogeneous of degree one in the input prices
- Nondecreasing in q, v, and w
- Concave in input prices

Cost Functions Are Concave in Input Prices



With input prices w' and v' , total costs of producing q_0 are C (v', w', q_0). If the firm does not change its input mix, costs of producing q_0 would follow the straight line C_{PSEUDO}. With input substitution, actual costs C (v', w, q_0) will fall below this line, and hence the cost function is concave in w.

Contingent Demand

- Input demand: firm’s demand on input given input prices
- Obtained from Shepherd’s lemma

EXAMPLE 10.4 Contingent Input Demand Functions

- Fixed proportions, $C(v,w,q)=q(v/\alpha+w/\beta)$
  - Contingent demand functions are quite simple:

$$k^c(v,w,q) = \frac{\partial C(v,w,q)}{\partial v} = \frac{q}{\alpha}$$

$$l^c(v,w,q) = \frac{\partial C(v,w,q)}{\partial w} = \frac{q}{\beta}$$

- Cobb-Douglas:

$$C(v,w,q) = vk + wl = q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{\beta/\alpha+\beta}$$

  - Contingent demand functions:

$$k^c(v,w,q) = \frac{\partial C}{\partial v} = \frac{\alpha}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B v^{-\beta/\alpha+\beta} w^{\beta/\alpha+\beta}$$

$$= \frac{\alpha}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B \left(\frac{w}{v}\right)^{\beta/\alpha+\beta}$$

$$l^c(v,w,q) = \frac{\partial C}{\partial w} = \frac{\beta}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B v^{\alpha/\alpha+\beta} w^{-\alpha/\alpha+\beta}$$

$$= \frac{\beta}{\alpha+\beta} \cdot q^{1/\alpha+\beta} B \left(\frac{w}{v}\right)^{-\alpha/\alpha+\beta}$$

EXAMPLE 10.4 Contingent Input Demand Functions

- CES:

$$C(v,w,q) = q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)}$$

  - The contingent demand functions:

$$k^c(v,w,q) = \frac{\partial C}{\partial v} = \frac{1}{1-\sigma}\cdot q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)}(1-\sigma)v^{-\sigma}$$

$$= q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} v^{-\sigma}$$

$$l^c(v,w,q) = \frac{\partial C}{\partial w} = \frac{1}{1-\sigma}\cdot q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)}(1-\sigma)w^{-\sigma}$$

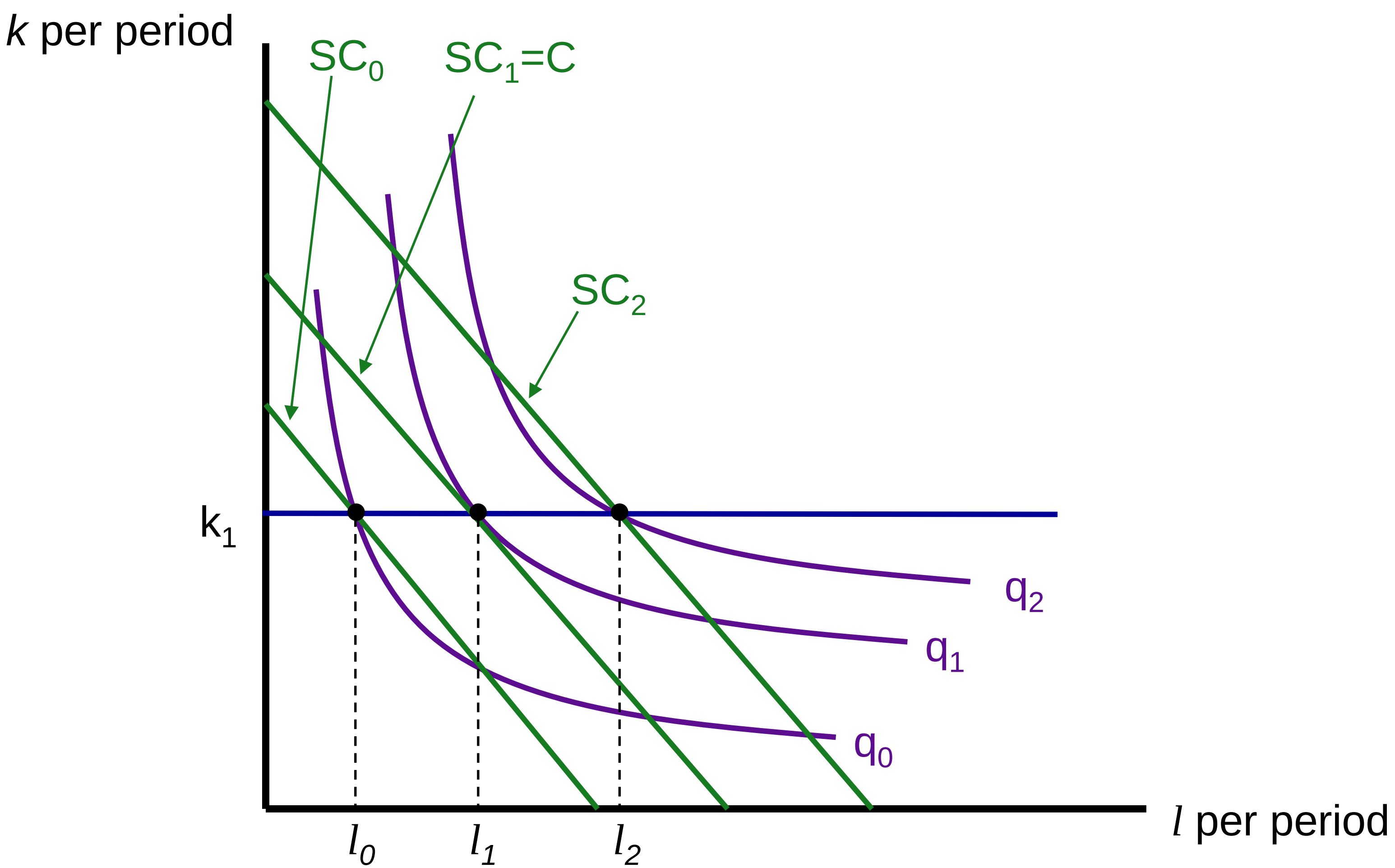
$$= q^{1/\gamma}\left(v^{1-\sigma} + w^{1-\sigma}\right)^{\sigma/(1-\sigma)} w^{-\sigma}$$

Short run and Long run

- Short-run: one factor is fixed
- Long-run: all factors can be varied
- Assume: capital is held constant at k_1
  - The firm is free to vary only its labor input
  - The production function is $q = f(k_1, l)$
  - Short run total cost: $SC = vk_1 + wl$
  - Short-run average cost: $SAC = SC/q$
  - Short-run marginal cost: $SMC = \partial SC/\partial q$

FIGURE 10.7

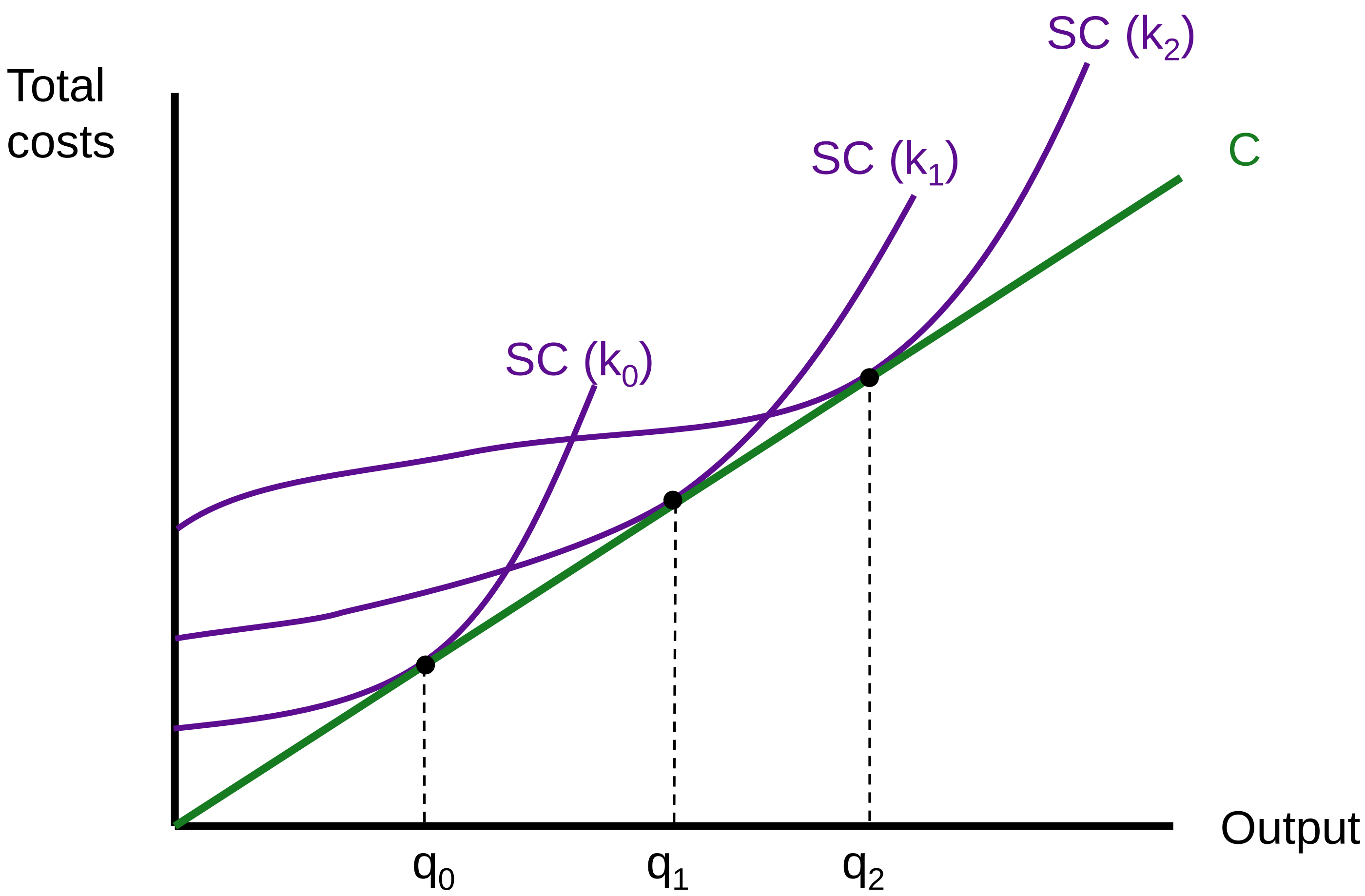
"Nonoptimal" Input Choices Must Be Made in the Short Run



Because capital input is fixed at k, in the short run the firm cannot bring its *RTS* into equality with the ratio of input prices. Given the input prices, q_0 should be produced with more labor and less capital than it will be in the short run, whereas q_2 should be produced with more capital and less labor than it will be.

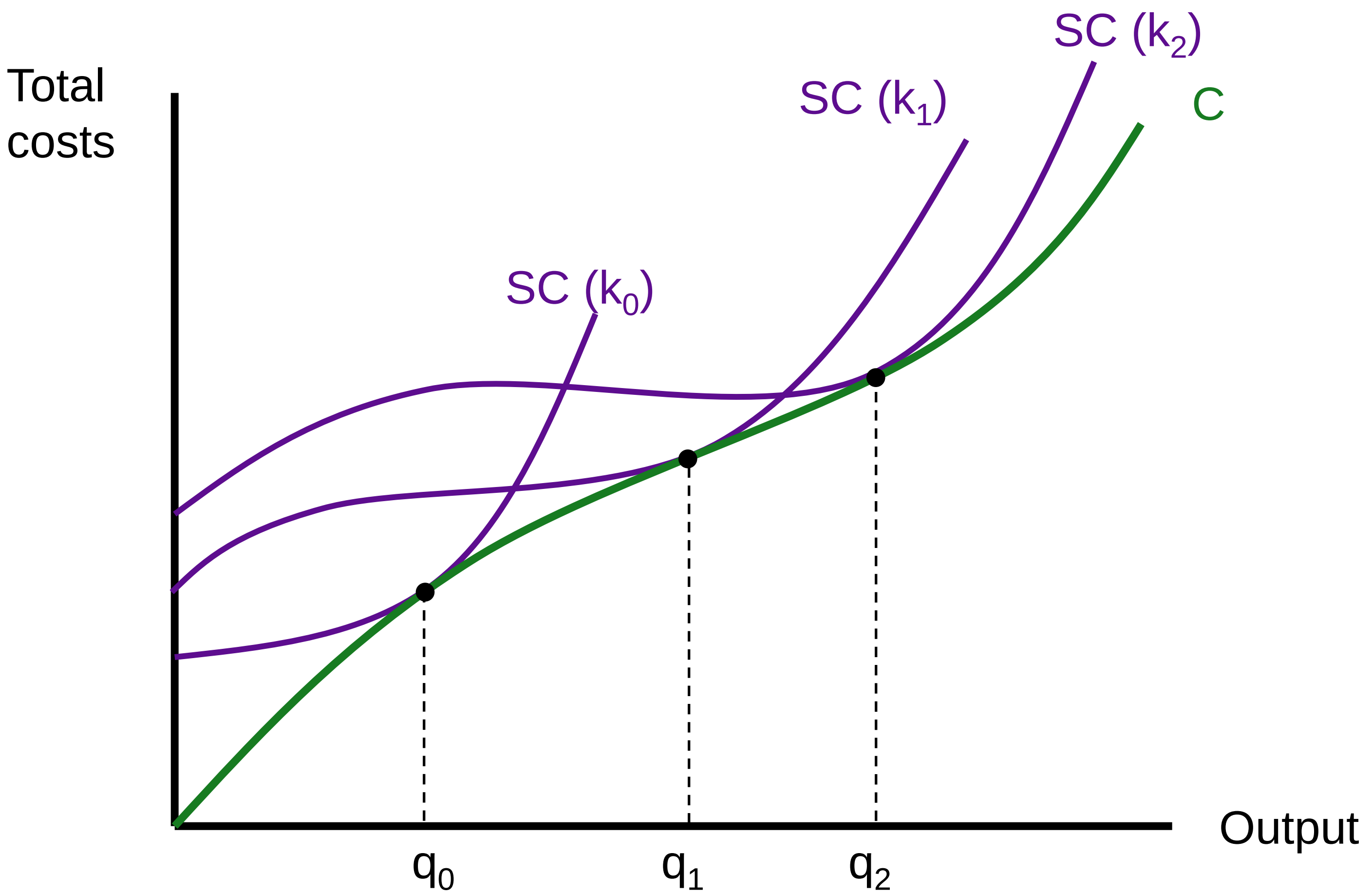
FIGURE 10.8 (a) Constant returns to scale

Two Possible Shapes for Long-Run Total Cost Curves



By considering all possible levels of capital input, the long-run total cost curve (C) can be traced. In (a), the underlying production function exhibits constant returns to scale: In the long run, although not in the short run, total costs are proportional to output.

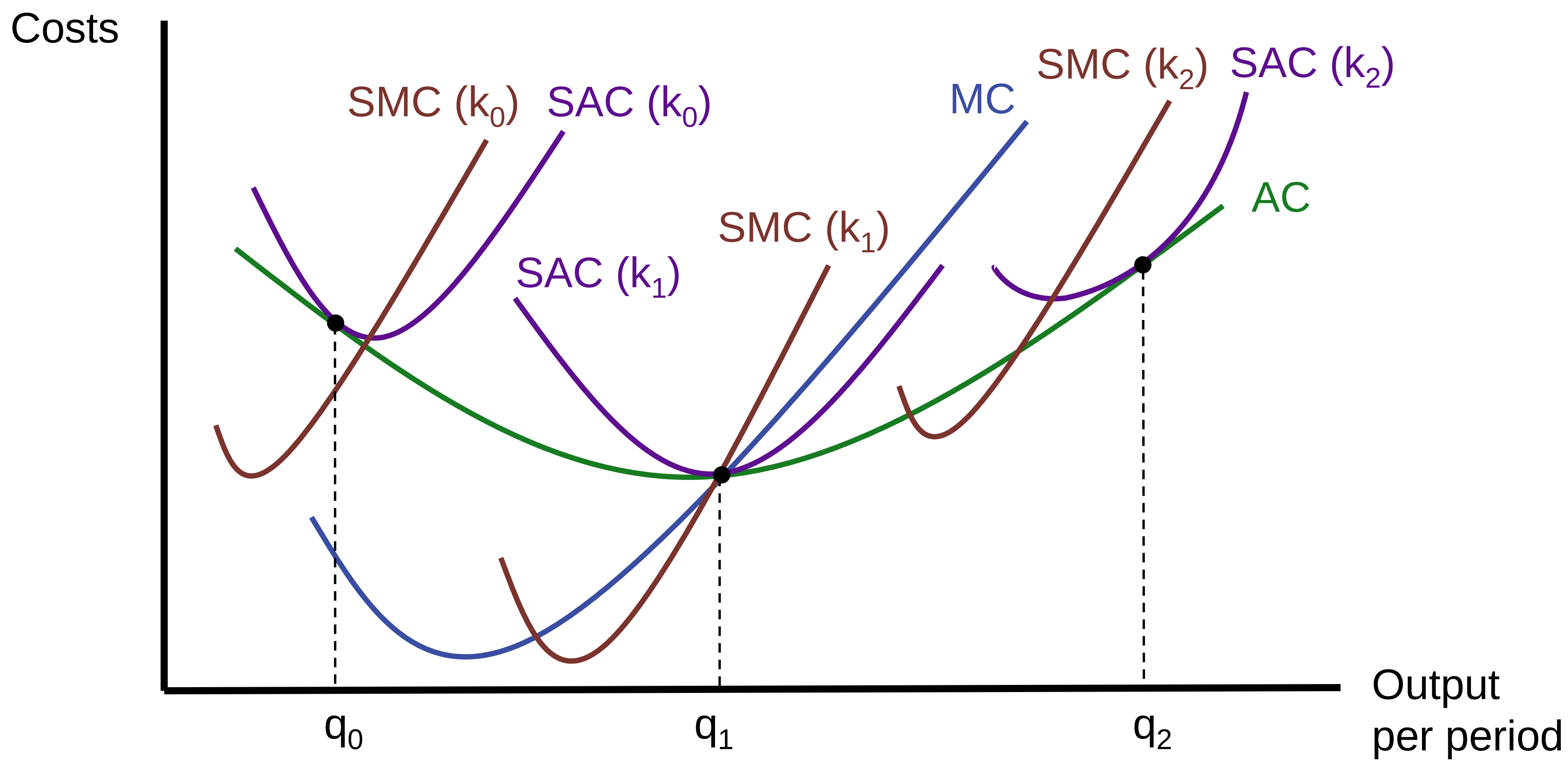
Two Possible Shapes for Long-Run Total Cost Curves



By considering all possible levels of capital input, the long-run total cost curve (C) can be traced. In (b), the long-run total cost curve has a cubic shape, as do the short-run curves. Diminishing returns set in more sharply for the short-run curves, however, because of the assumed fixed level of capital input.

FIGURE 10.9

Average and Marginal Cost Curves for the Cubic Cost Curve Case



This set of curves is derived from the total cost curves shown in Figure 10.8. The AC and MC curves have the usual U-shapes, as do the short-run curves. At q_1, long-run average costs are minimized. The configuration of curves at this minimum point is important.

Short-Run and Long-Run Costs

- At the minimum point of the *AC* curve:
  - The *MC* curve crosses the *AC* curve
    - *MC* = *AC* at this point
  - The *SAC* curve is tangent to the *AC* curve
    - *SAC* (for this level of *k*) is minimized at the same level of output as *AC*
    - *SMC* intersects *SAC* also at this point

$$AC = MC = SAC = SMC$$

The Translog cost function

- The translog function with two inputs

$$\ln C(q,v,w) = \ln q + a_0 + a_1 \ln v + a_2 \ln w + a_3 (\ln v)^2 +$$
$$+ a_4 (\ln w)^2 + a_5 \ln v \ln w$$

  - Implicitly assumes constant returns to scale
  - Homogeneous of degree 1 in input prices if: $a_1 + a_2 = 1$ and $a_3 + a_4 + a_5 = 0$
  - Includes the Cobb–Douglas as the special case $a_3 = a_4 = a_5 = 0$

The Translog cost function

- The translog function with two inputs
  - Input shares – easy to compute:

$$s_i = (\partial \ln C)/(\partial \ln w_i)$$

  - Elasticity of substitution

$$e_{k^c,w} = \frac{\partial \ln C_v}{\partial \ln w} = s_l + \frac{\alpha_5}{s_k}$$

  - Allen elasticity of substitution

$$A_{kl}=1+\alpha_5/s_k s_l$$

The Translog cost function

- Many-input translog cost function
  - n inputs, each with a price of w_i(i = 1,…, n)

$$C(q, w_1, ..., w_n) = \ln q + a_0 + \sum_{i=1}^{n} a_i \ln w_i +$$

$$+0.5 \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \ln w_i \ln w_j, \quad \text{where} \quad a_{ij} = a_{ji}$$

  - Constant returns to scale

The Translog cost function

- Many-input translog cost function
  - Homogeneous of degree 1 in the input prices if $\sum_{i=1}^{n} a_i = 1$ and $\sum_{i=1}^{n} a_{ij} = 0$
  - Input shares take the linear form

$$s_i = a_i + \sum_{j=1}^{n} a_{ij} \ln w_j$$

  - Elasticity of substitution between any two inputs

$$s_{ij} = 1_i + \frac{s_j a_{ij} - s_i a_{jj}}{s_i s_j}$$

Profit Maximization

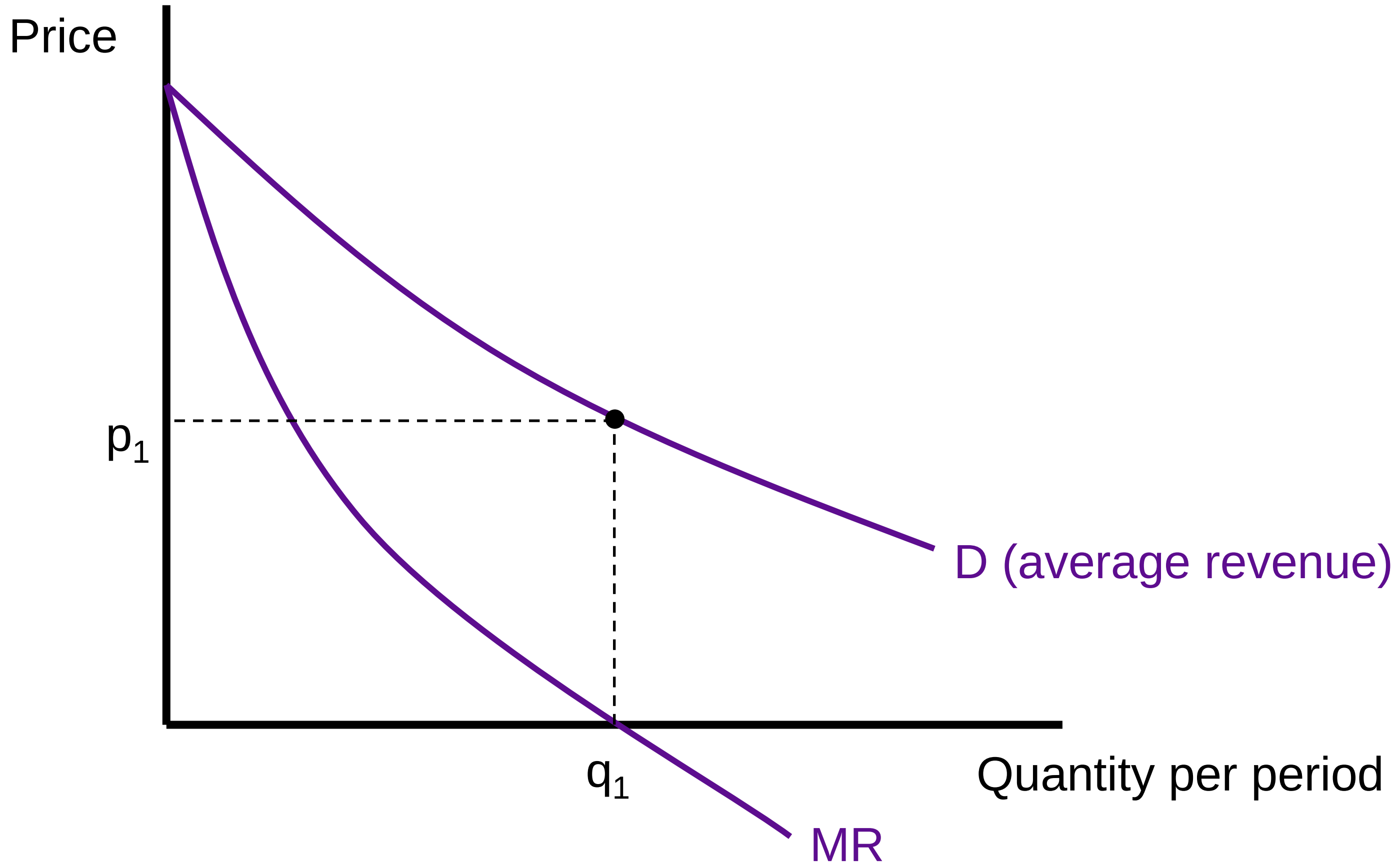
Revenue

- Revenue: R(q) = p(q)q
- Marginal revenue:

$$MR(q) = \frac{dR}{dq} = \frac{d[p(q) \cdot q]}{dq} = p + q \cdot \frac{dp}{dq}$$

- If dp/dq=0, firm is a price-taker.
  – MR(q) = p(q)
- If dp/dq<0,
  – MR(q) < p(q): MR curves is lower than p(q) curve

Market Demand Curve and Associated Marginal Revenue Curve



Because the demand curve is negatively sloped, the marginal revenue curve will fall below the demand (“average revenue”) curve. For output levels beyond q_1, MR is negative. At q_1, total revenues ($p_1 \cdot q_1$) are a maximum; beyond this point, additional increases in q cause total revenues to decrease because of the concomitant decreases in price.

EXAMPLE 11.2 The Constant Elasticity Case

- Demand function of the form: $q = ap^b$
  - Has a constant price elasticity of demand = $-b$
  - Solving this equation for p, we get

    $p = (1/a)^{1/b}q^{1/b} = kq^{1/b}$ where $k = (1/a)^{1/b}$
  - Hence: $R = pq = kq^{(1+b)/b}$
  - And $MR = dr/dq = [(1+b)/b]kq^{1/b} = [(1+b)/b]p$
- This implies that MR is proportional to price

Profit Maximization

- Profit: $\pi(q) = R(q) - C(q) = p(q) \cdot q - C(q)$
- FOC (necessary): MR = MC

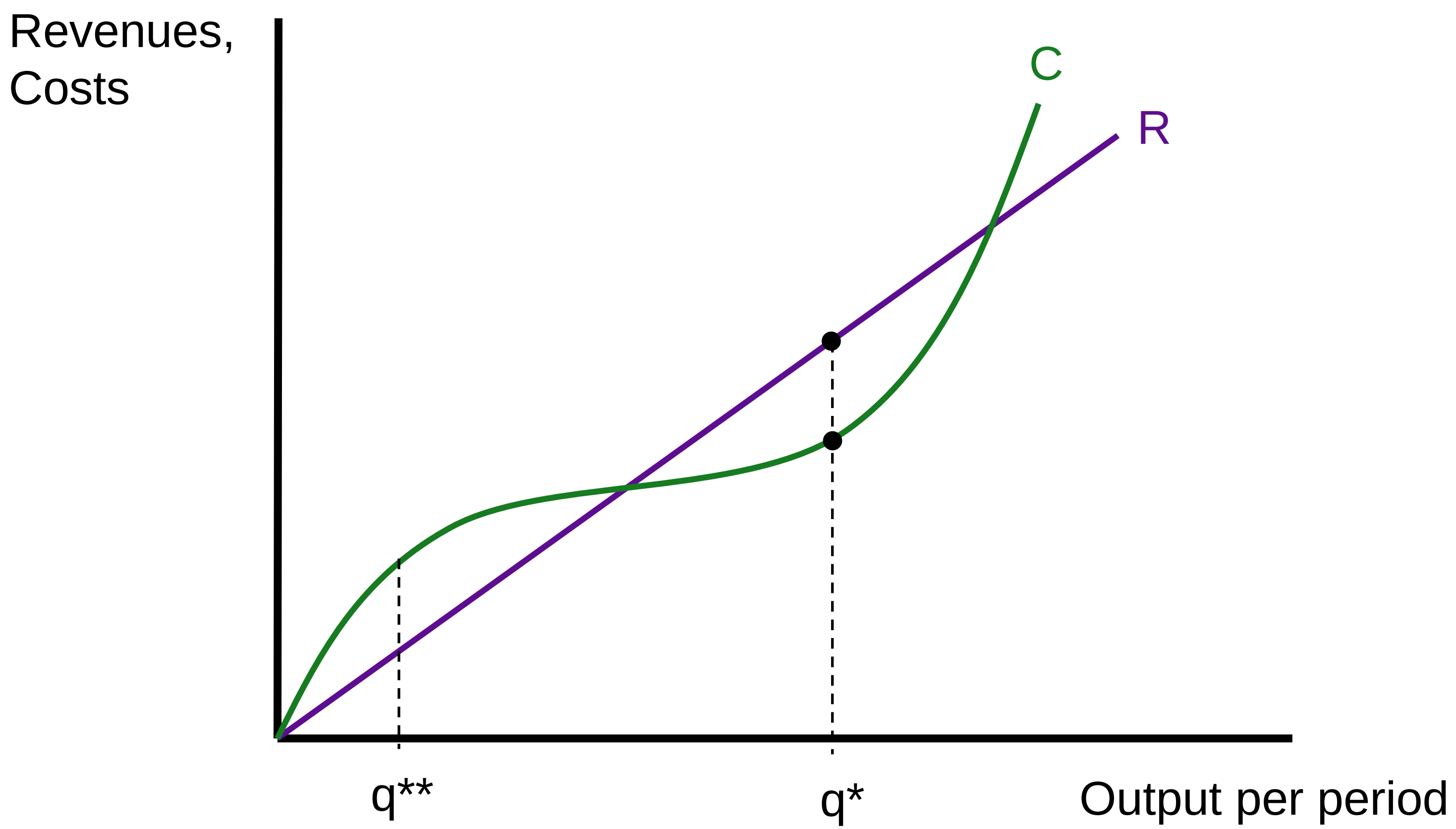
$$\frac{d\pi}{dq} = \pi'(q) = \frac{dR}{dq} - \frac{dC}{dq} = 0$$

$$\frac{dR}{dq} = \frac{dC}{dq}$$

- SOC (sufficient):
  - For q<q*, π'(q) > 0
  - For q>q*, π'(q) < 0

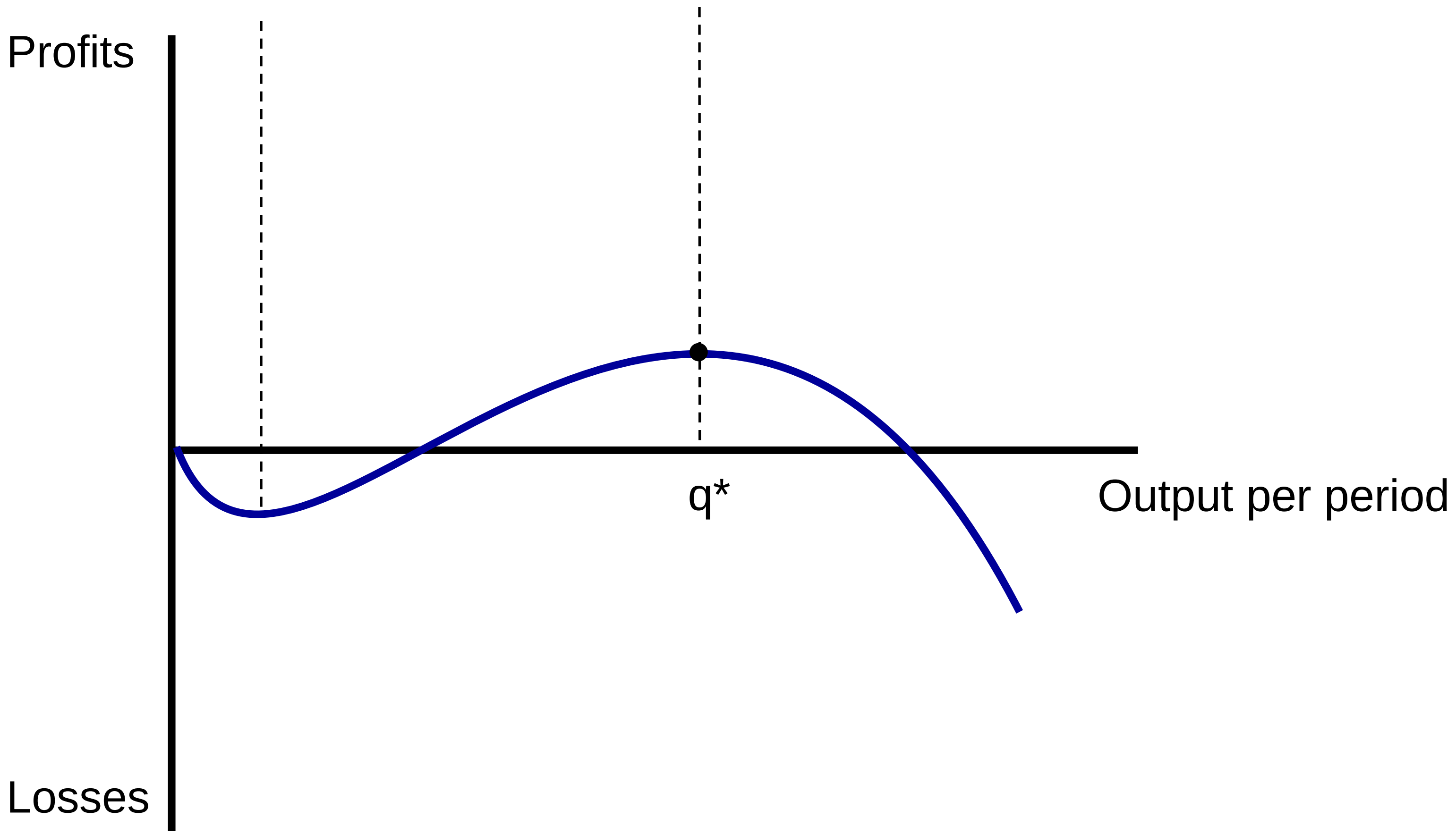
$$\left.\frac{d^2\pi}{dq^2}\right|_{q=q^*} = \left.\frac{d\pi'(q)}{dq}\right|_{q=q^*} < 0$$

Marginal Revenue Must Equal Marginal Cost for Profit Maximization



Profits, defined as revenues (R) minus costs (C), reach a maximum when the slope of the revenue function (marginal revenue) is equal to the slope of the cost function (marginal cost). This equality is only a necessary condition for a maximum, as may be seen by comparing points q* (a true maximum) and q** (a local minimum), points at which marginal revenue equals marginal cost.

Marginal Revenue Must Equal Marginal Cost for Profit Maximization



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EXAMPLE 11.1 Marginal Revenue from a Linear Demand Function

- Demand curve for a sub sandwich is

$$q = 100 - 10p$$

- Solving for price: $p = -q/10 + 10$
- Total revenue: $R = pq = -q^2/10 + 10q$
- Marginal revenue: $MR = dR/dq = -q/5 + 10$
  - MR < p for all values of q
- If the average and marginal costs are constant ($4)
  - Profit maximizing quantity: MR = MC, so q*=30
  - Price = $7, and profits = $90

Elasticity

- price elasticity of demand
  - Percentage change in quantity that results from a one percent change in price

$$e_{q,p} = \frac{dq/q}{dp/p} = \frac{dq}{dp} \cdot \frac{p}{q}$$

- Related to Marginal revenue

$$MR = p + \frac{q \cdot dp}{dq} = p\left(1 + \frac{q}{p} \cdot \frac{dp}{dq}\right) = p\left(1 + \frac{1}{e_{q,p}}\right)$$

Elasticity and MR

- Infinitely elastic: $e_{q,p} = -\infty$ and MR = p
- Elastic demand: $e_{q,p} < -1$ and MR > 0
- Unit Elastic demand: $e_{q,p} = -1$ and MR = 0
- Inelastic demand: $e_{q,\, p} > 1$ and MR < 0

Markup

- FOC: MR = MC

$$MC = p\left(1+\frac{1}{e_{q,p}}\right), \quad \text{or} \qquad \frac{p-MC}{p} = \frac{1}{-e_{q,p}} = \frac{1}{\left|e_{q,p}\right|}$$

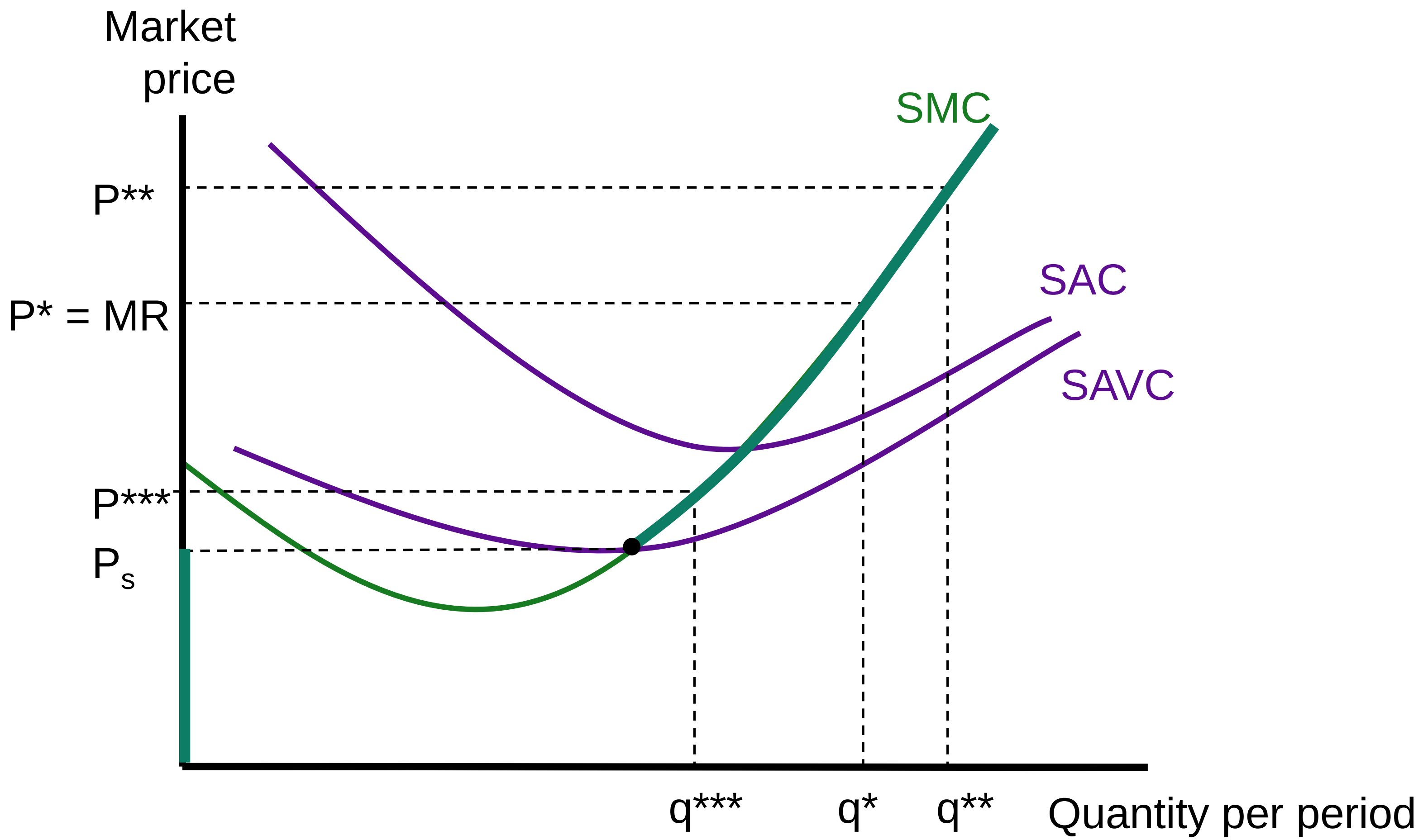
- Inverse elasticity rule
  - More elastic demand implies lower markup
  - Firm chooses elastic portion of demand curve

Short-run Supply of a price-taking firm

- Price-taking: P*=MR
- Profit: P>SAC ; Loss: P<SAC
- Operate: P>SAVC; Shutdown: P< SAVC
- Short-run supply curve: SMC above SAVC

FIGURE 11.3

Short-Run Supply Curve for a Price-Taking Firm



In the short run, a price-taking firm will produce the level of output for which SMC =P. At P*, for example, the firm will produce q*. The SMC curve also shows what will be produced at other prices. For prices below SAVC, however, the firm will choose to produce no output. The heavy lines in the figure represent the firm's short-run supply curve.

- Firm's short-run total cost curve is

$$SC(v,w,q,k) = vk_1 + wq^{1/\beta}k_1^{-\alpha/\beta}$$

  - Where k_1 is the level of capital held constant in the short run

- Short-run marginal cost is

$$SMC(v,w,q,k_1) = \frac{\partial SC}{\partial q} = \frac{w}{\beta} q^{(1-\beta)/\beta} k_1^{-\alpha/\beta}$$

- The price-taking firm will maximize profit where $p = SMC$

$$SMC = \frac{w}{\beta} q^{(1-\beta)/\beta} k_1^{-\alpha/\beta} = P$$

EXAMPLE 11.3 Short-Run Supply

- Quantity supplied will be

$$q = \left(\frac{w}{\beta}\right)^{-\beta/(1-\beta)} k_1^{-\alpha/(1-\beta)} P^{\beta/(1-\beta)}$$

- To find the firm's shut-down price, we need to solve for *SAVC*

$$SVC = wq^{1/\beta}k_1^{-\alpha/\beta}$$

$$SAVC = SVC/q = wq^{(1-\beta)/\beta}k_1^{-\alpha/\beta}$$

- *SAVC* < *SMC* for all values of $\beta < 1$
  - There is no price low enough that the firm will want to shut down

Profit Function

- Profit function: given product and factor prices

$$\Pi(P, v, w) = \max_{k,l} \pi(k, l) = \max_{k,l}[Pf(k, l) - vk - wl]$$

- First-order conditions:

$$\partial\pi/\partial k = P[\partial f/\partial k] - v = 0$$
$$\partial\pi/\partial l = P[\partial f/\partial l] - w = 0$$

  - Marginal revenue product: marginal contribution to revenues of hiring extra unit of input

$$MRP_l = Pf_l \text{ and } MRP_k = Pf_k$$

  - Hence, MRP = factor price
  - Implies cost minimization: *RTS* = *w*/*v*

Profit Function

- Second-order conditions:

$$\pi_{kk} = f_{kk} < 0$$

$$\pi_{ll} = f_{ll} < 0$$

$$\pi_{kk}\,\pi_{ll} - \pi_{kl}^{2} = f_{kk}f_{ll} - f_{kl}^{2} > 0$$

Input Demand functions

- Depends on factors on product prices

Capital Demand = $k(P,v,w)$

Labor Demand = $l(P,v,w)$

- Directly: optimal solution to profit function
- Indirectly: Envelope Theorem

$$\frac{\partial \Pi(P,v,w)}{\partial v} = -k(P,v,w)$$

$$\frac{\partial \Pi(P,v,w)}{\partial w} = -l(P,v,w)$$

Profit Function

- Properties:
  - homogeneous of degree one in all prices
  - Nondecreasing in output price, P
  - Nonincreasing in input prices, v, and w
  - Convex in output prices

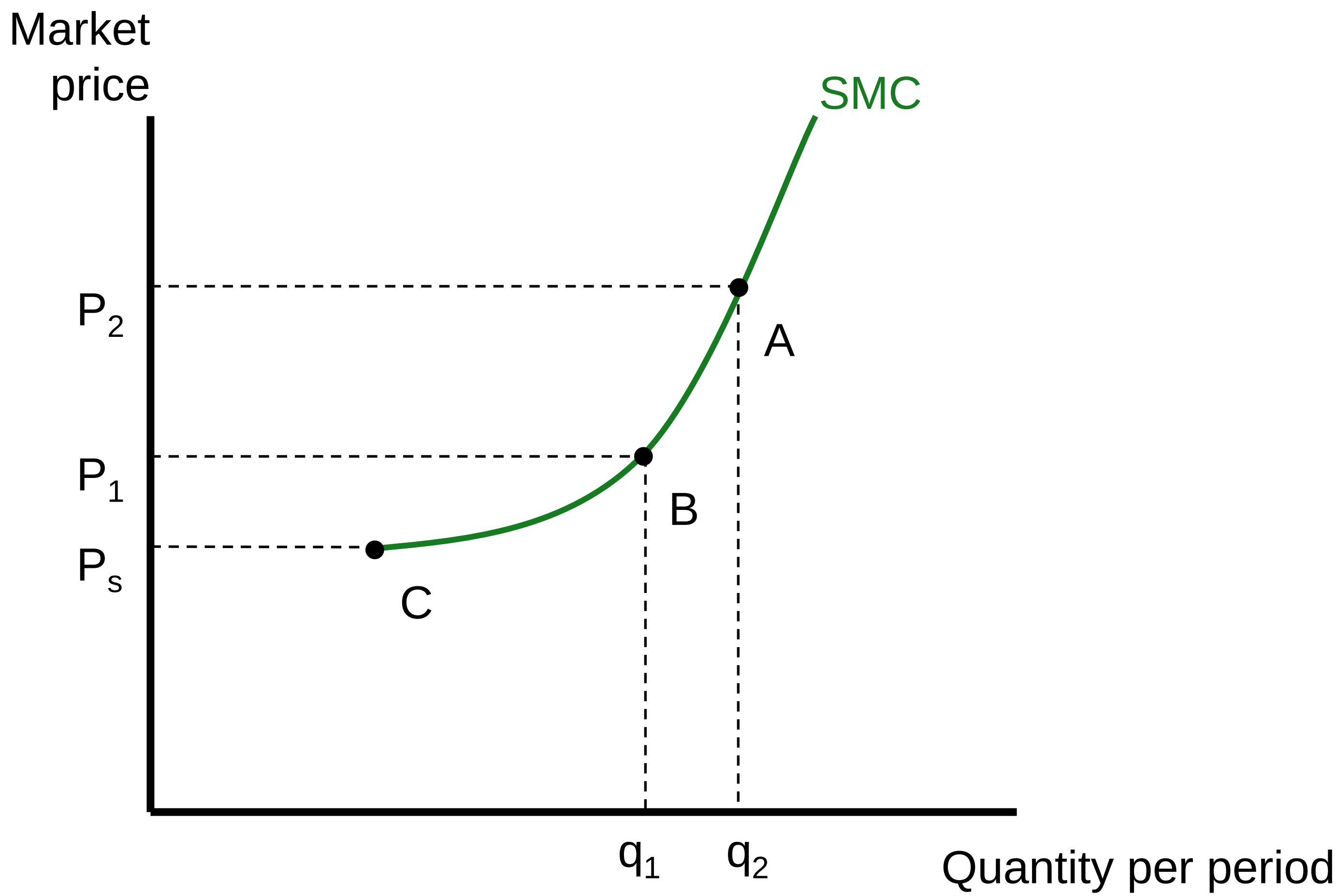
$$\frac{\Pi(P_1, v, w) + \Pi(P_2, v, w)}{2} \geq \Pi\left[\frac{P_1 + P_2}{2}, v, w\right]$$

Producer Surplus

- Producer surplus: profit
- Welfare gain for price goes up from P_1 to P_2:

$$\text{welfare gain} = \Pi(P_2,\ldots) - \Pi(P_1,\ldots) =$$
$$= \int_{p_1}^{p_2} \frac{\partial \Pi}{\partial P} dP = \int_{P_1}^{P_2} q(P) dP$$

Changes in Short-Run Producer Surplus Measure Firm Profits



If price increases from P_1 to P_2, then the increase in the firm's profits is given by area P_2ABP_1. At a price of P_1, the firm earns short-run producer surplus given by area P_sCBP_1. This measures the increase in short-run profits for the firm when it produces q_1 rather than shutting down when price is P_s or below.

EXAMPLE 11.4 A Short-Run Profit Function

- Cobb–Douglas production function, $q=k^{\alpha}l^{\beta}$
  - With $k=k_1$ in the short-run
- To find the profit function
  - Use the first-order conditions for a maximum

$$\frac{\partial \pi}{\partial l} = \beta P k_1^{\alpha} l^{\beta-1} - w = 0, \quad \text{so} \quad l = \left(\frac{w}{\beta P k_1^{\alpha}}\right) 1/(\beta-1)$$

$$\text{with } A = (w / \beta P k_1^{\alpha})$$

$$\Pi(P, v, w, k_1) = \frac{1-\beta}{\beta^{\beta/(\beta-1)}} w^{\beta/(\beta-1)} p^{1/(1-\beta)} k_1^{\alpha/(1-\beta)} - v k_1$$

Single Input: Only labor

- First-order condition

$$F(l,w,p)=Pf_t\text{-}w=0$$

- Implicit function

$$\frac{dl}{dw}=\frac{-\partial F/\partial w}{\partial F/\partial l}=\frac{w}{Pf_{ll}}\leq 0$$

Two inputs case

- Conditional demand for labor, $l^c(v,w,q)$
- Unconditional demand for labor, $l(P,v,w)$
- At the profit-maximizing level of output

$$l(P,v,w) = l^c(v,w,q) = l^c(v,w,\, q(P,v,w))$$

- Differentiation with respect to *w* yields

$$\frac{\partial l(P,v,w)}{\partial w} = \overbrace{\underbrace{\frac{\partial l^c(v,w,q)}{\partial w}}_{\text{substitution effect}} + \underbrace{\frac{\partial l^c(v,w,q)}{\partial q} \cdot \frac{\partial q}{\partial w}}_{\text{output effect}}}^{}$$

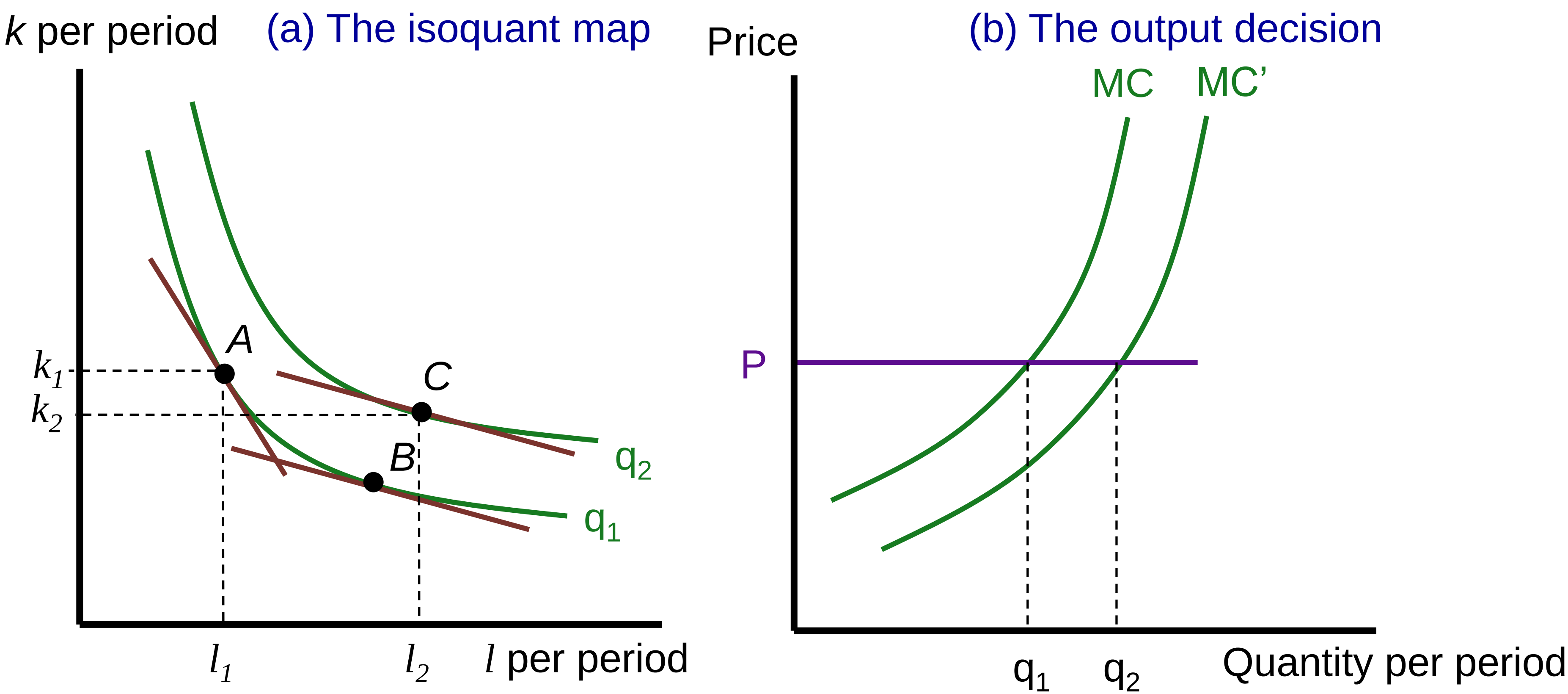
total effect

Substitution and Output Effects

- When w falls
  - Substitution effect
    - If output is held constant, there will be a tendency for the firm to want to substitute l for k in the production process
  - Output effect
    - A change in w will shift the firm's expansion path
    - The firm's cost curves will shift and a different output level will be chosen

FIGURE 11.5

The Substitution and Output Effects of a Decrease in the Price of a Factor



When the price of labor falls, two analytically different effects come into play. One of these, the substitution effect, would cause more labor to be purchased if output were held constant. This is shown as a movement from point A to point B in (a). At point B, the cost-minimizing condition ($RTS = w/v$) is satisfied for the new, lower w. This change in w/v will also shift the firm's expansion path and its marginal cost curve. A normal situation might be for the MC curve to shift downward in response to a decrease in w as shown in (b). With this new curve (MC') a higher level of output (q_2) will be chosen. Consequently, the hiring of labor will increase (to l_2), also from this output effect.

EXAMPLE 11.5 Decomposing Input Demand into Substitution and Output Components

- Cobb–Douglas function
  - When one of the inputs is held fixed

$$q = f(k,l,g) = k^{0.25}l^{0.25}g^{0.5}$$

  - Features:
    1. Permits capital–labor substitution
    2. Exhibits increasing marginal costs
  - Where
    - k is capital input
    - l is labor input
    - g is the size of the factory, held constant at 16

EXAMPLE 11.5 Decomposing Input Demand into Substitution and Output Components

- Cobb–Douglas function: $q = 4\,k^{0.25}l^{0.25}$
  - The factory can be rented at a cost of *r* per square meter per period
- Total cost function

$$C(v,w,r,q) = \frac{q^2 v^{0.5} w^{0.5}}{8} + 16r$$

- Profit function

$$\Pi(P,v,w,r) = 2P^2 v^{-0.5} w^{-0.5} - 16r$$

EXAMPLE 11.5 Decomposing Input Demand into Substitution and Output Components

- Envelope results
  - A change in the wage has a larger effect on total labor demand
  - Than it does on contingent labor demand
  - Because the exponent of w is more negative in the total demand equation

$$l^c(v,w,r,q) = \frac{\partial C}{\partial w} = \frac{q^2 v^{0.5} w^{-0.5}}{16}$$

$$l(P,v,w,r) = \frac{\partial \Pi}{\partial w} = P^2 v^{-0.5} w^{-0.5}$$

EXTENSIONS

Boundaries of the firm

- Common features of alternative theories
  - Property rights theory
  - Transactions cost theory
  - Factors: uncertainty, complexity, and specialization

Property rights theory

- $S(x_F, x_G)$
  - Total surplus generated by the transaction between Fisher Body and GM
    - Bargaining: each firm receives half of the surplus
  - The sum of both firm's profits
- x_F
  - Investments made by Fisher Body
- x_G
  - Investments made by GM

Property rights theory

- Efficient investment levels
  - Maximize total surplus minus investment costs

$$S(x_F,x_G) - x_F - x_G$$

- First-order conditions

$$\frac{\partial S}{\partial x_F} = \frac{\partial S}{\partial x_G} = 1$$

Property rights theory

- Objective function: $0.5\ S(x_F,x_G)$ - x_F
  - First-order condition
    - Investments by the firms (if they are two separate firms)

    $0.5\ (\partial S/\partial x_F)=1$ and $0.5\ (\partial S/\partial x_G)=1$
- GM acquires Fisher Body
  - They become one firm
  - Objective function: $0.5\ S(x_F,x_G)$ - x_G
  - First-order condition: $\partial S/\partial x_G=1$

Transactions cost theory

- h_F
  - Costly action undertaken by Fisher Body at the time of bargaining - increases its bargaining power at the expense of GM
  - Haggling
- h_G
  - Costly action undertaken by GM at the time of bargaining - increases its bargaining power at the expense of Fisher Body
  - Haggling

Transactions cost theory

- Bargaining shares
  - $\alpha(h_F,h_G)$ - share accruing to Fisher Body
  - $1 - \alpha(h_F,h_G)$ - share accruing to GM
  - Where $0 < \alpha < 1$, increasing in h_F and decreasing in h_G
- Efficient levels of investment: x_F* and x_G*

Transactions cost theory

- Fisher Body's objective function
  - Determining its equilibrium level of haggling

$$\alpha(h_F,h_G)\,[S(x_F^*,x_G^*)- x_F^* - x_G^*]-h_F$$

- First-order conditions

$$\frac{\partial \alpha}{\partial x_F}[S(x_F^*,x_G^*)-x_F^*-x_G^*]=1$$

$$\frac{\partial \alpha}{\partial x_G}[S(x_F^*,x_G^*)-x_F^*-x_G^*]=1$$