An Individual's Utility Max. Problem: An Analytical Example (1/7)

Let $U(c_1, c_2) = \ln(c_1) + \beta \ln(c_2)$.

Individual's maximization problem is

$$\max_{c_1, c_2} U(c_1, c_2) \;\; s.t. \;\; c_1 + \frac{c_2}{1+r} = \omega$$

Using the trick of expressing c_2 in terms of c_1, his maximization problem is now reduced to one unknown c_1

$$\max_{c_1} \ln(c_1) + \beta \ln((\omega - c_1)(1+r))$$

An Individual's Utility Max. Problem: An Analytical Example (2/7)

Hence, differentiate objective function with respect to c_1 directly:

$$F.O.C.\,(c_1):\ \frac{1}{c_1^*}+\beta\frac{-(1+r)}{\left(\omega-c_1^*\right)(1+r)}=0,$$

which can be manipulated to yield

$$c_1^*=\frac{\omega}{1+\beta}. \tag{6}$$

Since in equilibrium,

$$c_2^*=(\omega-c_1^*)(1+r), \tag{7}$$

An Individual's Utility Max. Problem: An Analytical Example (3/7)

substituting (6) into (7) :

$$c_2^* = \frac{(1+r)\beta}{1+\beta}\omega.$$

Lastly, to solve for s_1^*, we know that the first period budget constraint must be satisfied in equilibrium:

$$s_1^* = y_1 - c_1^* = \frac{1}{1+\beta}\left(\beta y_1 - \frac{y_2}{1+r}\right).$$

An Individual's Utility Max. Problem: An Analytical Example (4/7)

Another method:

$$\begin{aligned} & \max_{c_1} \ln(c_1) + \beta \ln(c_2(c_1)) \\ where\ c_2(c_1) &= (1+r)(\omega - c_1) \\ F.O.C. &: \frac{c_2^*}{\beta c_1^*} = 1+r, \end{aligned}$$

where if we multiply both sides by (-1) :

$$\underbrace{-\frac{c_2^*}{\beta c_1^*}}_{MRS_{c_1,c_2}} = \underbrace{-(1+r)}_{\text{slope of LBC}} \text{ which is just } \underbrace{-\frac{u'(c_1^*)}{\beta u'(c_2^*)}}_{MRS_{c_1,c_2}} = \underbrace{-(1+r)}_{\text{slope of LBC}}$$

An Individual's Utility Max. Problem: An Analytical Example (5/7)

Manipulating the F.O.C.:

$$c_2^* = \beta(1+r)c_1^* \tag{8}$$

In equilibrium, $\left(c_1^*, c_2^*\right)$ must satisfy the LBC, that is,

$$c_1^* + \frac{c_2^*}{1+r} = \omega, \tag{9}$$

so we can substitute (8) into (9) and manipulate to yield

$$c_1^* = \frac{\omega}{1+\beta}. \tag{10}$$

An Individual's Utility Max. Problem: An Analytical Example (6/7)

Now substitute (10) into (8) :

$$c_2^* = \frac{(1+r)\beta}{1+\beta}\omega.$$

Lastly, to solve for s_1^*, we know that the first period budget constraint must be satisfied in equilibrium:

$$s_1^* = y_1 - c_1^* = \frac{1}{1+\beta}\left(\beta y_1 - \frac{y_2}{1+r}\right).$$

An Individual's Utility Max. Problem: An Analytical Example (7/7)

F.O.C. gives us the tangency of IC to LBC, which picks out the point of the maximum utility a consumer can obtain given his lifetime budget constraint.

If the consumer is maximizing his utility, consumer is choosing $\left(c_1^*, c_2^*\right)$ in such a way that the F.O.C. is satisfied.

Note that even though we reduced the maximization problem to a problem in terms of c_1, it's a trick to simplify the problem, but we must remember that c_2 and s_1 are still equilibrium objects we have to solve for!

Optimal Consumption Path (1/2)

Define the **discount rate** ρ as $\beta = \frac{1}{1+\rho}$. Our usual F.O.C.

$$\frac{u'(c_1^*)}{\beta u'(c_2^*)} = 1 + r \tag{5}$$

can now be expressed as

$$\frac{u'(c_1^*)}{u'(c_2^*)} = \frac{1+r}{1+\rho}. \tag{11}$$

Since utility function u has these properties: $u' > 0$, $u'' < 0$:

$$\begin{aligned} u'(c_1^*) &> u'(c_2^*) \text{ if and only if } c_1^* < c_2^* \\ u'(c_1^*) &< u'(c_2^*) \text{ if and only if } c_1^* > c_2^* \\ u'(c_1^*) &= u'(c_2^*) \text{ if and only if } c_1^* = c_2^* \end{aligned}$$

Optimal Consumption Path (2/2)

So we have three cases

Case 1:	$r > \rho$	$c_1^* < c_2^*$
Case 2:	$r < \rho$	$c_1^* > c_2^*$
Case 3:	$r = \rho$	$c_1^* = c_2^*$

(3 different consumption profiles: figure 5)

Case 3: interest rate exactly compensates individual for his impatience;
Case 2: interest rate does not compensate for his impatience, so he consumes more in first than second period:
Case 1, relatively patient and interest rate more than compensates for his degree of impatience, so consume less today than tomorrow.

Remark

What is the key message to take out of individual's max. problem thus far?
- Individual **consumes out of lifetime wealth**, not just current income

Since an economy is composed of many, many agents, each agent is making optimal consumption choices each period based on what their lifetime wealth is. Aggregate consumption is thus an aggregation of each individual's consumption choices.

Comparative Statics: Increase in Income (1/5)

Note: assume consumption goods are **normal** goods

Question: By how much do would an individual react to changes in income?
Answer: Depends on whether the change in income is temporary or permanent.

Let us first focus on temporary increases in income.

In fact, the individual's Marginal Propensity to Consume (MPC) to consume out of current income less than one (consumption smoothing)

Let me convince you of this through an example

Comparative Statics: Increase in Income (2/5)

Example: $u(c_1) = \ln c_1$, $u(c_2) = \ln c_2$.

Recall that

$$c_1^* = \frac{\omega}{1+\beta}, c_2^* = \frac{\beta\,(1+r)\,\omega}{1+\beta}, s_1^* = \frac{1}{1+\beta}\left(\beta y_1^* - \frac{y_2^*}{1+r}\right)$$

How do consumption decisions respond to a change in y_1?

$$\frac{\partial c_1^*}{\partial y_1} = \frac{1}{1+\beta} \in (0,1)\,,\ \frac{\partial c_2^*}{\partial y_1} = \frac{\beta\,(1+r)}{1+\beta} \in (0,1)\,, \frac{\partial s_1^*}{\partial y_1} = \frac{\beta}{1+\beta} \in (0,1)$$

Hence c_1^* changes **less than proportionally** with a change in first period income. Hence, if y_1 increases by one unit, then c_1^* increases by $\frac{1}{1+\beta} \in (0,1)$ unit, and s_1^* increases by $(1 - \frac{1}{1+\beta}) = \frac{\beta}{1+\beta} \in (0,1)$ units

Comparative Statics: Increase in Income (3/5)

- individual is "spreading out" increase in current income over both periods, i.e., he is **smoothing consumption**

- hence, he is able to increase c_2^*. By what fraction? Since he is saving $\frac{\beta}{1+\beta}$ units in the first period which earns him interest, in the second period, he can consume an additional $\frac{\beta}{1+\beta}(1+r)$

Clear from this example that MPC out of current income less than one

See Figure 6 for graphical analysis of temporary changes in income:
either $\Delta y_1 > 0$, $\Delta y_2 = 0$,
or $\Delta y_2 > 0$, $\Delta y_1 = 0$.

Comparative Statics: Increase in Income (4/5)

Now suppose that both periods' incomes rise: $\Delta y_1 = \Delta y_2 > 0$.

It is as if your income has increased permanently, so call it a permanent change in income

From the lectures, we know that consumption choices are made out of lifetime wealth. So in deciding how to change consumption choices when income changes, either temporary or permanent, we basically have to ask how much wealth changes in response to these changes

Comparative Statics: Increase in Income (5/5)

To do so, we introduce the notion of **Permanent Income Hypothesis (Milton Friedman)**

Message: consumption depends on lifetime wealth (closely related to permanent income), but changes in temporary income yield smaller changes in lifetime wealth (permanent income) which affects consumption by less than permanent changes in income (which yield larger changes in lifetime wealth, and permanent income).

See Figure 7

Comparative Statics: Increase in r (1/5)

Question: How does consumption change in both periods when r increases?
Answer: Depends on whether you are a borrower or lender

Two effects:

1. Substitution effect
2. Income effect

Substitution effect:

- Consuming today is more expensive relative to consuming tomorrow, because the opportunity cost of consuming today has increased (one unit of savings yields a higher return when r is higher); price of consuming today relative to tomorrow is $(1+r)$

Comparative Statics: Increase in r (2/5)

- $\left(\frac{1}{1+r}\right)$ is the price of consumption goods tomorrow in terms of current consumption goods. Increase in r means that $\left(\frac{1}{1+r}\right)$ is lower, i.e., cheaper to consume tomorrow.

$\Rightarrow$ consume less today and more tomorrow, or $c_1^* \downarrow$, $s_1^* \uparrow$ and $c_2^* \uparrow$

Substitution effect is the same for both borrowers and lenders

BUT income effect works in opposite directions depending on whether you are a borrower or lender

Income effect:
If borrower, feel poorer as pay back more interest on loan; $c_1^* \downarrow$, $s_1^* \uparrow$ and $c_2^* \downarrow$
If lender, feel richer because get more interest on loan; $c_1^* \uparrow$, $s_1^* \downarrow$ and $c_2^* \uparrow$

Comparative Statics: Increase in r (3/5)

Summarizing results for a lender:

	c_1^*	s_1^*	c_2^*
substitution effect	$\downarrow$	$\uparrow$	$\uparrow$
income effect	$\uparrow$	$\downarrow$	$\uparrow$

and for a borrower,

	c_1^*	s_1^*	c_2^*
substitution effect	$\downarrow$	$\uparrow$	$\uparrow$
income effect	$\downarrow$	$\uparrow$	$\downarrow$

Comparative Statics: Increase in r (4/5)

Effect of an interest rate increase on LBC? (figure 8)
- recall that LBC's slope is $-(1+r)$
- intercepts of LBC? When $c_2 = 0$, $c_1 = \omega$ (PV of lifetime earnings),
- and when $c_1 = 0$, $c_2 = \omega(1+r)$ (future value of lifetime earnings)

Let r be the old interest rate, $\widetilde{r}$ be the new interest rate, $\widetilde{r} > r$
- new LBC's slope if $-(1+\widetilde{r}) < -(1+r)$ (new LBC is steeper)

$$\begin{aligned} \omega &\equiv y_1 + \frac{y_2}{1+r} \ (LBC) \\ \omega' &\equiv y_1 + \frac{y_2}{1+\widetilde{r}} < \omega, \text{ and} \end{aligned}$$

$$\begin{aligned} \omega(1+r) &= y_1(1+r) + y_2 \\ \omega'(1+\widetilde{r}) &= y_1(1+\widetilde{r}) + y_2 > \omega(1+r) \end{aligned}$$

Comparative Statics: Increase in r (5/5)

See figure 9

- For aggregate consumption function we will assume that the substitution effect dominates the income effect so that current consumption decreases with rises in the interest rate
  - backed by Macro data (savings increase with r increases)

Definition of Competitive Equilibrium (1/4)

A competitive equilibrium for the economy considered here with N individuals is an allocation $\left\{c_1^{i*}, c_2^{i*}, s_1^{i*}\right\}_{i=1}^{N}$, giving the amount consumed and saved in each period by each individual i, and an interest rate r^*, such that:

- for each individual i, taking the interest rate r^* as given, $\left(c_1^{i*}, c_2^{i*}, s_1^{i*}\right)$ is the solution to his lifetime utility maximization problem;

- the credit market clears at the interest rate r^*, i.e., total borrowing = total savings, or

$$S_1^P\left(r^*\right) = 0, \tag{12}$$

  where $S_1^P\left(r\right)$ is the aggregate amount of (private) savings at time t when the interest rate is r.

Definition of Competitive Equilibrium (2/4)

Why "competitive"?
Because all prices (real interest rate) taken as given when agents are optimizing, implying that each agent considers that his actions have no impact on the market interest rate. Hence, this notion of competitive equilibrium only makes sense when the population of agents in the economy is large.

Interpretation of equation (12): the real interest rate will adjust to clear the credit market.

There is an alternative way to express this market clearing condition in equation (12). Denote Y_t as aggregate output, and C_t as aggregate consumption.

Definition of Competitive Equilibrium (3/4)

Since

$$\begin{aligned} S_t^P(r^*) &= Y_t - C_t = 0 \\ &\Longrightarrow Y_t = C_t,\ t \in \{1,2\} \end{aligned} \tag{13}$$

then (13) is an alternative way of expressing the credit market clearing condition

This alternative way is: that goods market clears.

Walras' Law tells us that in our economy, if credit market clears, goods market clears; and goods market clears, credit market clears. That is why we only need to write the condition that one of the two markets here clears.

Definition of Competitive Equilibrium (4/4)

Note that credit and goods markets are intricately linked. If goods demanded in goods market exceeds goods supplied, then it must be because too few individuals want to save relative to the number who want to borrow. Conversely, if goods supplied exceeds goods demanded in the goods market, then there are relatively too few borrowers. It is clear that when demand equals supply in both markets, then both markets clear. But we know from Walras's Law that if there are M markets in the economy and if $(M-1)$ markets clear, than the last market must clear. Which is why we can focus on credit market clearing. At the market clearing interest rate r^*, $S_t^P(r^*) = 0$, credit market clears, and goods market must also be clearing.