Urban Economics: Economics 4621

University of Minnesota, Spring 2004

N11: Notes on Congestion

MODEL WITH FIXED CAPACITY

Suppose that there is a highway that connects the suburbs to the downtown. Let Q denote the number of drivers on this road. The time cost per driver $A(Q)$ (the average time cost) depends upon the number of drivers. Assume $A'(Q) = 0$ for $Q \leq X$ (where X is referred to as capacity) and $A'(Q) > 0$ for $Q > X$. This captures the idea that when there are few cars on the road, the addition of another car makes no difference for congestion. But after a certain point, addition of cars begins to congest the highway and commute times goes up.

Total time cost is

$$T(Q) = A(Q)Q.$$

and marginal time cost is

$$M(Q) = T'(Q) = A(Q) + A'(Q)Q.$$

The marginal time cost equals the average time cost plus the change in average cost multiplied by the number of drivers.

Let $D(Q)$ be note the inverse demand for driving when prices are denoted in units of time. This is the marginal willingness to pay (in time) for the Qth driver.

The equilibrium level of driving Q^e is the point where the private marginal benefit equals the private cost

$$D(Q^e) = A(Q^e)$$

as illustrated in the figure. But note that at that point, the social marginal cost $M(Q)$ exceeds the private cost $A(Q)$. The socially efficient level Q^* is the point

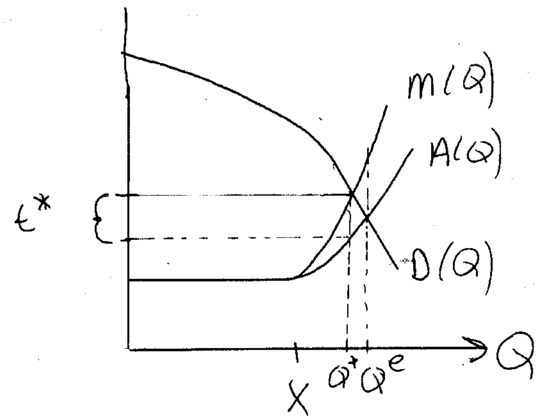

FIG. 1.

where the social marginal benefit (here equal to private marginal benefit) is equal to social marginal cost (here greater than the private marginal cost),

$$D(Q^*) = M(Q^*) = A(Q^*) + A'(Q^*)Q^*.$$

To obtain the socially efficient outcome, a congestion tax of

$$t^* = A'(Q^*)Q^*$$

can be imposed. With this tax, a driver will internalize the effects of his or her commute in raising the time cost of others and the efficient outcome is obtained.

VARIABLE CAPACITY

Now allow for changes in capacity. Write the average cost function as $A(Q, X)$ to depend upon capacity. Assume the following functional form:

$$\begin{aligned} A(Q, X) &= a,\ Q \leq X \\ &= a + b\left(\frac{Q}{X} - 1\right)^2,\ Q > X. \end{aligned}$$

Then total cost and marginal cost (for $Q > X$) is

$$\begin{aligned} T(Q,X) &= aQ + b\left(\frac{Q}{X} - 1\right)^2 Q \\ M(Q,X) &= \frac{\partial T}{\partial Q} = a + b\left(\frac{Q}{X} - 1\right)^2 + 2b\left(\frac{Q}{X} - 1\right)\frac{Q}{X}. \end{aligned}$$

Suppose that the cost of capacity is c per unit of capacity. Assume the units are in time, the same units as for the $A(Q, X)$ function.

Consider a social planner picking Q^* and X^* to maximize total surplus. Like above, the choice of Q must set the social marginal benefit of one more driver equal to the social marginal cost for fixed $X = X^*$; i.e.

$$\begin{aligned} D(Q^*) &= M(Q^*, X^*) \\ &= a + b\left(\frac{Q^*}{X^*} - 1\right)^2 + 2b\left(\frac{Q^*}{X^*} - 1\right)\frac{Q^*}{X^*} \end{aligned} \tag{1}$$

The choice of X must minimize total cost given Q^*. Total cost TC equals total driving cost plus cost of capacity

$$\begin{aligned} TC &= T(Q,X) + cX \\ &= aQ + b\left(\frac{Q}{X} - 1\right)^2 Q + cX. \end{aligned}$$

The first-order condition for the choice of X is

$$\frac{\partial TC}{\partial X} = -2b\left(\frac{Q^*}{X^*} - 1\right)\frac{Q^{*2}}{X^{*2}} + c = 0$$

We can rewrite the above condition and get

$$2b\left(\frac{Q^*}{X^*} - 1\right)\frac{Q^*}{X^*} = \frac{cX^*}{Q^*}$$

Observe that the right-hand side of the above is the average cost of capacity. (It equals total capacity cost divided by the number of drivers). The left-hand side is equal to the $A'(Q)Q$ term, i.e., the third term on the right-hand side of equation (1). This is the externality term. Thus if a congestion tax is set equal to

$$t^* = 2b\left(\frac{Q^*}{X^*} - 1\right)\frac{Q^*}{X^*} = \frac{cX^*}{Q^*},$$

the first best is obtained. In summary, when capacity is variable and when there are constant returns to scale, the optimal congestion tax t^* equals the average cost of capacity cX^*/Q^*.