N4: Monocentric Model Part 2

Remaining Conditions For Equilibrium

- $\hat{u}$ is boundary of city

$$R(\hat{u}) = \overline{R}$$

- Supply equals demand for land

$$\int_0^{\hat{u}} \frac{1}{L(u)} du = H$$

- What is $S = D$ formula if land is a plane not a line and city is a circle?

Summary of Conditions for Equilibrium

- Set of objects:

—$\hat{u}$ city boundary

—$R(u)$ rent on land at distance $u \in [0, \hat{u}]$,

—$x(u), L(u)$,: non-land and land consumption of individual at location u

- That satisfy:

1. $(x(u), L(u))$ maximizes $U(x, L)$ at u, given budget constraint

$$px + R(u)L = w - tu$$

2. Individuals are indifferent to any location $u \in [0, \hat{u}]$. Or

$$R'(u) = -\frac{t}{L(u)}$$

3. $R(\hat{u}) = \overline{R}$

4. Supply equals Demand. Or

$$\int_0^{\hat{u}} \frac{1}{L(u)} du = H$$

Comparative Statics

- *Within city.* How does land consumption vary with u?

- *Same city over time*

—Effect of an increase in H

—Effect of a decrease in t

—Analysis of other changes for homework

Within City: How does $L(u)$ vary with u?

- Budget constraint: $w - ut$. So income falls with u
- Opportunity cost of one more unit of land in terms of widgets,
$$\frac{R(u)}{p}$$
- Price of land $R(u)$,
$$R'(u) = -\frac{t}{L(u)} < 0$$
so opportunity cost of land falls with u.
- Pure substitution effect (since U constant)

- Graph
- So $L(u)$ strictly increases
- $L'(u) > 0$ implies $R''(u) > 0$, i.e. R convex
- Density $D(u) = \frac{1}{L(u)}$,strictly declines in u. (Recall density gradient)

Same City over Time: Effect of Increase in Population H

- Let H_1 and H_2 be initial and new populations, $H_1 < H_2$.
- Let $R_1(\cdot)$ and $R_2(\cdot)$ and initial and new rent functions
- Starting from initial equilibrium, which of the four conditions is no longer satisfied at the new, higher population level?

- Answer: Condition 4, supply equals demand for land, is no longer satisfied.
- Claim: the rent function must shift up at every point, i.e. $R_2(u) > R_1(u)$, $u < \hat{u}_2$.
- Proof. Suppose rent functions intersect...(graphical argument on whiteboard)...Get contradiction

Conclusion

- The rent function shifts up
- The boundary of the city $\hat{u}$ expands
- Density $D(u) = \frac{1}{L(u)}$ increases

Effect of Increase in t

- Suppose t_1 is initial level and new $t_2 < t_1$.
- Assume land a normal good (an assumption on $U(x, L)$)
- Two-step strategy.

(1) pivot rent function to intermediate value $\tilde{R}(\cdot)$ to get conditions 1, 2, and 3, to hold

(2) then shift to get supply equals demand condition to hold.

Step 1: Pivot $R(\cdot)$ around $\hat{u}_1$

- Compare locations $\hat{u}_1$ and 0 at original price
- Income at $\hat{u}_1, I = w - t\hat{u}_1$ is higher with lower transportation cost, but at 0 income is the same.
- So price of land must fall at $u = 0$ to retain indifference.
- Construct $\tilde{R}(\cdot)$ so that $\tilde{R}(\hat{u}_1) = \overline{R}$ and conditions 1 and 2 hold.

.

Step 2: Shift $\tilde{R}(\cdot)$

- What about supply and demand at $\tilde{R}(\cdot)$ and new transportation cost?

- Answer: $\tilde{R}(u) < R_1(u)$, $u < \hat{u}_1$ (land prices lower) and $(w - t_1 u) > (w - t_2 u)$, $u > 0$ (incomes higher) implies demand for land increases at each location. (Here use assumption that land is a normal good.)

- So demand exceeds supply. We must shift $R(\cdot)$ up, $\hat{u}$ out, to get market clearing in land market.

- Claim: Even though it shifts up, land at the CBD is cheaper than before, $R_2(0) < R_1(0)$.

- Proof: With a decrease in t, equilibrium utility must strictly increase. This follows because an equilibrium here is Pareto Efficient (First Welfare Theorem), i.e. the same as the outcome of a social planner. If t decreases, welfare under a social planner increases, and also welfare under the market. Since the income at $u = 0$ remains the same at w, regardless of transportation costs, if $R_2(0) \geq R_1(0)$, utility at $u = 0$ would not increase, a contradiction.

.

Conclusion: Effect of a Decrease in t

- $\hat{u}_2 > \hat{u}_1$ (boundary extend further)

- $R_2(0) < R_1(0)$ (land prices fall at center)

- $R_2(\hat{u}_1) > R_1(\hat{u}_1) = \overline{R}$ (land prices rise further out)

- Average density $H/\hat{u}$ decreases

- $D_1(0) < D_2(0)$ (Density falls at center), and density gradient is flatter.

Extension: Individuals with Different Transportation Costs

- Suppose two types of people, 1 and 2, $t_1 > t_2$, but wage the same

- Conjecture the form of the equilibrium...

Proof

- Construct $R_1(u)$ to keep type 1 indifferent and $R_2(u)$ to keep type 2 indifferent

$$\begin{aligned} R_2'(u) &= -\frac{t_2}{L_2(u)} \\ R_1'(u) &= -\frac{t_1}{L1(u)} \end{aligned}$$

- Suppose at $\hat{u}_1$ these cross, then $R_1(\hat{u}_1) = R_2(\hat{u}_1)$. Since price of land is the same for both types at $\hat{u}_1$ and income is higher for type 2, $L_2(\hat{u}_1) > L_1(\hat{u}_1)$, so $R_1'(\hat{u}_1) < R_2'(\hat{u}_1)$ (using the fact that land is normal). So get complete sorting.

- Example: think of business at type 1 sector, residential use as the type 2 sector.